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LETTERS AND COMMENTS

Spherical volume averages of static electric and magnetic fields using Coulomb and Biot–Savart laws

Ben Yu-Kuang Hu

Department of Physics, University of Akron, Akron, OH 44325-4001, USA

E-mail: byhu@uakron.edu

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Abstract

Virtually identical derivations of the expressions for the spherical volume averages of static electric and magnetic fields are presented. These derivations utilize the Coulomb and Biot–Savart laws, and make no use of vector calculus identities or potentials.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The average of static electric or magnetic fields over a sphere has been used to obtain important results, such as the macroscopic electric field inside a dielectric [1] and the presence of δ-function electric (magnetic) fields at the position of electric (magnetic) point dipoles [2]. In both electric and magnetic cases, if the sources of the fields are outside the averaging sphere, the average field equals the value of the field at the centre of the sphere, whereas for sources inside the averaging sphere, the average electric and magnetic fields are proportional to the electric and magnetic dipole moments of the sources, respectively.

Textbooks typically only treat the electric field case [3]. Only a handful of authors [2, 4] treat both electric and magnetic cases, probably because the magnetic field case is considered to be 'tough' by undergraduate standards[1], since the derivations typically employ the magnetic vector potential and vector calculus identities. This paper presents derivations of both the electric and magnetic field cases that are virtually identical and are elementary, in the sense that they rely on Coulomb's and the Biot–Savart laws and make no use of vector calculus identities or potentials.

[1] Reference [1], pp 156–7 and p 253, problem 5.57.

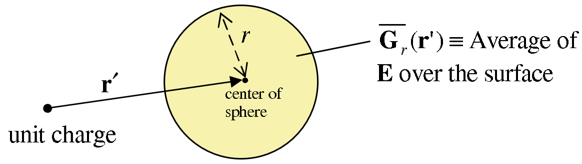

Figure 1. Definition of $\overline{\mathbf{G}_r}(\mathbf{r}')$: the average of the electric field of a unit charge about a spherical surface of radius r, centred at $\mathbf{r}'$ relative to the charge.

Coulomb's law states that the electric field for a unit point charge is

$$\mathbf{G}(\mathbf{r}) = k\frac{\mathbf{r}}{r^3}, \tag{1}$$

where $\mathbf{r}$ is a displacement vector from the point charge, $r = |\mathbf{r}|$ and $k = (4\pi\epsilon_0)^{-1}$ and 1 in SI and Gaussian units, respectively. In terms of $\mathbf{G}$, the electric field $\mathbf{E}$ at point $\mathbf{s}$ due to a static charge distribution $\rho(\mathbf{r}')$ is

$$\mathbf{E}(\mathbf{s}) = \int \mathrm{d}\mathbf{r}'\rho(\mathbf{r}')\,\mathbf{G}(\mathbf{s}-\mathbf{r}') = -\int \mathrm{d}\mathbf{r}'\rho(\mathbf{r}')\,\mathbf{G}(\mathbf{r}'-\mathbf{s}). \tag{2}$$

Define $\overline{\mathbf{G}_r}(\mathbf{r}')$ to be the *average of* $\mathbf{G}$ *over a spherical surface of radius r around a point* $\mathbf{r}'$, as shown in figure 1. An apparently little-known result that is critical in the following discussion is (see the appendix for derivations)

$$\overline{\mathbf{G}_r}(\mathbf{r}') = k\frac{\mathbf{r}'}{r'^3}\Theta(r'-r), \tag{3}$$

where Θ is the Heavyside step function[2]. Equation (3) implies that the average of the electric field over the surface of the sphere for a point charge outside the sphere ($r' > r$) equals the electric field at the centre of the sphere, whereas a point charge inside a sphere ($r' < r$) contributes *zero* to the average of the electric field over the surface of the sphere. In a sense, this result is the antithesis of the integral form of Gauss' law[3] but it is not as general as Gauss' law because it applies only to spherical surfaces.

2. Volume averages of static electric fields

Let $\langle\mathbf{E}\rangle_R$ to be average of the electric field $\mathbf{E}$ about a spherical *volume* of radius R and $\overline{\mathbf{E}_r}(\mathbf{0})$ be the average electric field over a spherical shell of radius r, both centred around the origin. Then,

$$\langle\mathbf{E}\rangle_R \equiv \frac{3}{4\pi R^3}\int_{r=0}^{R} \mathrm{d}\mathbf{r}\,\mathbf{E}(\mathbf{r}) = \frac{3}{R^3}\int_0^R \mathrm{d}r\, r^2\,\overline{\mathbf{E}_r}(\mathbf{0}). \tag{4}$$

For a given charge distribution $\rho(\mathbf{r}')$, $\overline{\mathbf{E}_r}(\mathbf{0})$ can be obtained by averaging both sides of (2) over a spherical shell given by $|\mathbf{s}| = r$, yielding

$$\overline{\mathbf{E}_r}(\mathbf{0}) = -\int \mathrm{d}\mathbf{r}'\rho(\mathbf{r}')\,\overline{\mathbf{G}_r}(\mathbf{r}'). \tag{5}$$

[2] The Heavyside function is defined to be $\Theta(x) = 1$ for $x > 0$, $\Theta(x) = 0$ for $x < 0$ and $\Theta(0) = \frac{1}{2}$.
[3] Note the difference between the average of a vector field $\mathbf{G}$ over a surface $\mathcal{A}$, $\int_{\mathcal{A}} \mathrm{d}A\,\mathbf{G}/|\mathcal{A}|$ (where $|\mathcal{A}|$ is the area of surface $\mathcal{A}$), and the flux of $\mathbf{G}$ through the surface, $\int_{\mathcal{A}} \mathrm{d}A\,\hat{\mathbf{n}}\cdot\mathbf{G}$ (where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the area element $\mathrm{d}A$). This paper deals with the former, while Gauss' law pertains to the latter.

2.1. Single point charge

We first examine the case of a single point charge. Because electric fields obey the principle of linear superposition, what holds for a single point charge is generalizable to cases of many charges and/or continuous charge distributions. Substituting the charge distribution for a point charge q at $\mathbf{d}$, $\rho(\mathbf{r}') = q\delta(\mathbf{r}' - \mathbf{d})$, into (5) gives $\overline{\mathbf{E}_r}(\mathbf{0}) = -q\overline{\mathbf{G}_r}(\mathbf{d})$. Substituting this into (4) and using (3) gives

$$\langle \mathbf{E} \rangle_R = -\frac{3kq\mathbf{d}}{d^3 R^3} \int_0^R \mathrm{d}r\, r^2 \Theta(d - r). \tag{6}$$

If the point charge is outside the averaging sphere of radius R, then $d > R$ and the integral in (6) equals $R^3/3$. Hence, $\langle \mathbf{E} \rangle_R^{(\mathrm{out})} = -kq\,\mathbf{d}/d^3 = \mathbf{E}(\mathbf{0})$, the electric field at the centre of the sphere due to the point charge. On the other hand, if the point charge is inside the averaging sphere, then $d < R$ and the integral in (6) equals $d^3/3$, in which case $\langle \mathbf{E} \rangle_R^{(\mathrm{in})} = -k\mathbf{p}/R^3$, where $\mathbf{p} = q\mathbf{d}$ is the dipole moment of the point charge relative to the centre of the sphere.

2.2. Arbitrary charge distribution

We now obtain these results more rigorously for an arbitrary charge distribution $\rho(\mathbf{r}')$. Substituting (3) and (5) into (4) yields,

$$\langle \mathbf{E} \rangle_R = -\frac{3}{R^3} \int_0^R \mathrm{d}r\, r^2 \left[\int_{r'=r}^{\infty} \mathrm{d}\mathbf{r}' \frac{k\rho(\mathbf{r}')\mathbf{r}'}{r'^3} \right], \tag{7}$$

where the effect of the Θ function in (3) is to restrict the $\mathbf{r}'$ integration to the region $|\mathbf{r}'| > r$. We now consider separately the contribution of the charges outside and inside the sphere.

2.2.1. Sources outside the sphere.

The contribution due to charges outside the sphere of radius R corresponds to $\int_{r'=R}^{\infty} \mathrm{d}\mathbf{r}'$ in (7), which makes the term in square parentheses independent of r. The integration over r gives $R^3/3$, yielding

$$\langle \mathbf{E} \rangle_R^{(\mathrm{out})} = -\int_{r'=R}^{\infty} \mathrm{d}\mathbf{r}' \frac{k\rho(\mathbf{r}')\mathbf{r}'}{r'^3} = \mathbf{E}^{(\mathrm{out})}(\mathbf{0}) \tag{8}$$

the electric field at the origin. Therefore, in general, $\langle \mathbf{E} \rangle_R^{(\mathrm{out})} = \mathbf{E}^{(\mathrm{out})}$ at the centre of the sphere.

2.2.2. Sources inside the sphere.

Sources inside the sphere of radius R correspond to $|\mathbf{r}'| < R$ in (7). This gives

$$\begin{aligned}
\langle \mathbf{E} \rangle_R^{(\mathrm{in})} &= -\frac{3}{R^3} \int_0^R \mathrm{d}r\, r^2 \int_{r'=r}^{R} \mathrm{d}\mathbf{r}' \frac{k\rho(\mathbf{r}')\mathbf{r}'}{r'^3} \\
&= -\frac{3k}{R^3} \int_{r'=0}^{R} \mathrm{d}\mathbf{r}' \frac{\rho(\mathbf{r}')\mathbf{r}'}{r'^3} \int_0^{r'} \mathrm{d}r\, r^2 \\
&= -\frac{k}{R^3} \int_{r'=0}^{R} \mathrm{d}\mathbf{r}' \rho(\mathbf{r}')\mathbf{r}' = -\frac{k\mathbf{p}}{R^3},
\end{aligned} \tag{9}$$

where $\mathbf{p} = \int \mathrm{d}\mathbf{r}' \rho(\mathbf{r}')\mathbf{r}'$ is the dipole moment relative to the centre of the sphere, and we have used $\int_0^R \mathrm{d}r \int_r^R \mathrm{d}r' = \int_0^R \mathrm{d}r' \int_0^{r'} \mathrm{d}r$, as shown in figure 2.

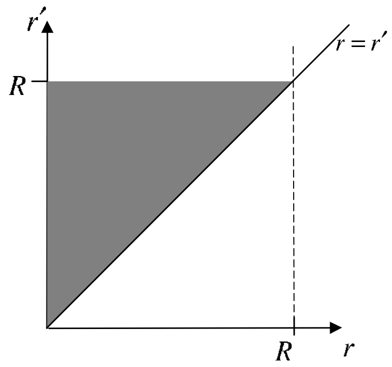

Figure 2. Integration region for the limits $\int_0^R \mathrm{d}r \int_r^R \mathrm{d}r'$ (grey region). Integration over this region can also be written as $\int_0^R \mathrm{d}r' \int_0^{r'} \mathrm{d}r$.

3. Magnetic field case

A static magnetic field $\mathbf{B}$ is related to the charge current density $\mathbf{J}$ by the Biot–Savart law,

$$\mathbf{B}(\mathbf{s}) = \int \mathrm{d}\mathbf{r}' \mathbf{J}(\mathbf{r}') \times \mathbf{G}(\mathbf{s} - \mathbf{r}'), \tag{10}$$

where $k = \mu_0/(4\pi)$ or c^{-1} for SI or Gaussian, respectively. Comparison of (10) with (2) shows that the derivation for the magnetic field case can be copied wholesale from the electric field case by substituting $\mathbf{E} \to \mathbf{B}$ and $\rho \to \mathbf{J}\times$ at every step. (The only point of caution is that the cross product anti-commutes, so care must be taken to preserve the order of $\mathbf{J}$ and $\overline{\mathbf{G}}$ or $\mathbf{r}'$.) Using these substitutions in (8) gives, for current sources outside the averaging sphere,

$$\langle \mathbf{B} \rangle_R^{(\mathrm{out})} = -\int_{r'=R}^{\infty} \mathrm{d}\mathbf{r}' \frac{k\mathbf{J}(\mathbf{r}') \times \mathbf{r}'}{r'^3} = \mathbf{B}(\mathbf{0}), \tag{11}$$

the magnetic field at the centre of the sphere. For current sources inside the averaging sphere, using the substitutions in (9) gives

$$\langle \mathbf{B} \rangle_R^{(\mathrm{in})} = -\frac{k}{R^3} \int_{r'=0}^{R} \mathrm{d}\mathbf{r}' \, \mathbf{J}(\mathbf{r}') \times \mathbf{r}' = \frac{2k\mathbf{m}}{R^3}, \tag{12}$$

where $\mathbf{m} = \frac{1}{2} \int \mathrm{d}\mathbf{r}' \, \mathbf{r}' \times \mathbf{J}(\mathbf{r}')$ is the magnetic dipole moment[4].

Appendix A. Derivation of (3)

By definition,

$$\overline{\mathbf{G}_r}(\mathbf{r}') = \frac{1}{4\pi r^2} \int \mathrm{d}A'' \, \mathbf{G}(\mathbf{r}''), \tag{A.1}$$

where the integral is over the spherical shell of radius r around $\mathbf{r}'$. Choose $\mathbf{r}' = r' \hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector in the z-direction. By azimuthal symmetry, $\overline{\mathbf{G}_r}(r'\hat{\mathbf{z}}) = \overline{G_{r,z}} \, \hat{\mathbf{z}} = (4\pi r^2)^{-1} \int \mathrm{d}A'' G_z(\mathbf{r}'')\hat{\mathbf{z}}$, where G_z is the z-component of $\mathbf{G}$. By dividing the spherical surface into strips $\mathrm{d}A'' = r^2 \sin\theta \, \mathrm{d}\theta$ (see figure 3), $\overline{G_{r,z}} = \frac{1}{2} \int_0^{\pi} \mathrm{d}\theta \sin\theta G_z(\mathbf{r}'')$. Letting $\mu = \cos\theta$,

[4] See, e.g. [1] p 254; [2] p 186.

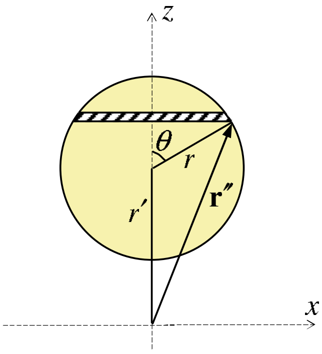

Figure 3. Spherical surface over which G_z is averaged to obtain $\overline{G_{r,z}}$. The hatched strip indicates the area element $\mathrm{d}A''$ on the sphere with constant $G_z(\theta) = kr''_z/|\mathbf{r}''|^3$, where $r''_z = r' + r\cos\theta$ and $|\mathbf{r}''| = \sqrt{r'^2 + r^2 + 2r\,r'\cos\theta}$.

and using $G_z(\mathbf{r}'') = kr''_z/|\mathbf{r}''|^3$ gives

$$\begin{aligned}\overline{G_{r,z}} &= \frac{k}{2}\int_{-1}^{1}\mathrm{d}\mu\,\frac{r\mu + r'}{(r^2 + r'^2 + 2r'r\mu)^{3/2}}\\ &= -\frac{k}{2}\frac{\partial}{\partial r'}\left[\int_{-1}^{1}\mathrm{d}\mu\,\frac{1}{(r^2 + r'^2 + 2r'r\mu)^{1/2}}\right]\\ &= -\frac{k}{2}\frac{\partial}{\partial r'}\left[\frac{\sqrt{r^2 + r'^2 + 2r'r\mu}}{r'r}\right]_{\mu=-1}^{1} = -\frac{k}{2}\frac{\partial}{\partial r'}\left[\frac{|r + r'| - |r - r'|}{r'r}\right]\\ &= \begin{cases} -k\dfrac{\partial r^{-1}}{\partial r'} = 0 & \text{for} \quad r' < r;\\ -k\dfrac{\partial r'^{-1}}{\partial r'} = \dfrac{k}{r'^2} & \text{for} \quad r' > r,\end{cases}\end{aligned} \tag{A.2}$$

which implies (3).

This result can also be obtained by using the well-known property that the average of an electrostatic potential of a point charge Q over the surface of a sphere of radius r equals the potential at the centre if the charge is outside the sphere, and kQ/r if the charge is inside[5]. Stated mathematically, if $V(\mathbf{r}'') = k/|\mathbf{r}''|$, then the average of V over a spherical shell of radius r around $\mathbf{r}'$ is

$$\overline{V_r}(\mathbf{r}') = k\begin{cases} r'^{-1} & \text{for} \quad |\mathbf{r}'| > r;\\ r^{-1} & \text{for} \quad |\mathbf{r}'| < r.\end{cases} \tag{A.3}$$

Averaging over spherical shells on both sides of the relation $\mathbf{G}(\mathbf{r}') = -\nabla' V(\mathbf{r}')$ yields $\overline{\mathbf{G}_r}(\mathbf{r}') = -\nabla'\overline{V_r}(\mathbf{r}')$. Using (A.3) in this gives (3).

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[2] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley) pp 148–50 and pp 187–8

[5] See, e.g. [1] pp 114–115.

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