

Density Dependent Exchange Contribution to $\partial\mu/\partial n$ and Compressibility in Graphene

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We calculate $\partial\mu/\partial n$ (where μ = chemical potential and n = electron density), which is associated with the compressibility, in graphene as a function of n , within the Hartree-Fock approximation. The exchange-driven Dirac-point logarithmic singularity in the quasiparticle velocity of intrinsic graphene disappears in the extrinsic case. The calculated renormalized $\partial\mu/\partial n$ in extrinsic graphene on SiO₂ has the same $n^{-(1/2)}$ density dependence but is 20% larger than the inverse bare density of states, a relatively weak effect compared to the corresponding parabolic-band case. We predict that the renormalization effect can be enhanced to about 50% by changing the graphene substrate.

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The band structure of graphene (a single layer of carbon atoms), by dint of its honeycomb lattice, has linear dispersions near the \mathcal{K} and \mathcal{K}' points (“Dirac points”) of the Brillouin zone. Recent developments in techniques for fabricating conducting graphene layers have thus provided the physics community with a unique opportunity to study an interacting two-dimensional (2D) massless Dirac fermion system using tabletop experimental equipment. This has led to a veritable explosion of both experimental and theoretical activity in this field [1].

Around the Dirac points (which we take to be the zero of energy), the kinetic energy for a “bare” electron (see below) is $\epsilon_{k,s}^{(0)} = s v_0 |\mathbf{k}|$, where \mathbf{k} is the wave vector with respect to the Dirac point, and $s = +1$ and -1 for the conduction and valence bands, respectively. The electron chemical potential μ , which in intrinsic graphene is at zero, can be shifted up or down by doping and/or application of external gate voltages, with a concomitant change in the electron density. This Letter reports the calculation of $\partial\mu/\partial n$, which is related to the electronic compressibility, in extrinsic graphene at temperature $T = 0$ as a function of the density n . [In this Letter, unless otherwise indicated, partial derivatives are at constant area and $T = 0$, and n refers exclusively to the *free* carrier density (i.e., the difference in electron density from that of intrinsic graphene) in the gated graphene, which we take to be substantially less ($|n| \sim 10^{12} \text{ cm}^{-2}$) than the intrinsic electron density n_v ($> 10^{15} \text{ cm}^{-2}$) filling up the valence band.] We obtain $\mu(n)$ by evaluating the electron self-energy within the Hartree-Fock approximation (HFA). The HFA is a good approximation up to reasonably high values of r_s (\sim the ratio of the average carrier potential to kinetic energy) in parabolic-band semiconductors, and we expect it to also give reliable results in graphene, where $r_s < 1$.

It is useful (and conceptually meaningful) to divide 2D graphene into three different systems depending on the band filling: *bare* or *empty*, a theoretical abstraction of

just one electron in the graphene honeycomb lattice as appropriate for the single-particle band-structure calculation with both valence and conduction bands completely (and unphysically) empty, or equivalently, the unphysical situation where the interaction between the electrons is turned off; *intrinsic*, i.e., the undoped and ungated situation, which is a zero-gap semiconductor with a completely full (empty) valence (conduction) band and chemical potential μ ($= E_F$, since we are at $T = 0$) precisely at the Dirac point; *extrinsic*, i.e., gated or doped graphene with a tunable 2D free carrier density n of electrons (holes) in the conduction (valence) band, with μ being above (below) zero, i.e., in the conduction (valence) band. Note that only the empty system can be characterized by the bare, noninteracting parameters (e.g., velocity v_0 , density of states D_0) with both intrinsic and extrinsic graphene being characterized by renormalized parameters. We emphasize that the bare graphene parameters, being unphysical abstractions, cannot be *experimentally* determined.

In the absence of interaction, $\partial\mu/\partial n$ is just the inverse of the bare or noninteracting single-particle density of states at the Fermi level: $(\partial n/\partial \mu)_0 \equiv D_0(E_F) = \sqrt{g n}/(\sqrt{\pi} \hbar v_0)$, where v_0 is the bare graphene carrier velocity associated with the linear energy dispersion, and g ($= 4$) is the product of the spin ($g_s = 2$) and valley ($g_v = 2$) degeneracies of the graphene carriers. This implies that $\partial\mu/\partial n$ in graphene is a direct measurement of the *thermodynamic* Fermi velocity renormalization due to electron-electron interaction effects. (This should be distinguished from the *quasiparticle* Fermi velocity renormalization, as discussed later.) Our goal here is to theoretically calculate the renormalized $\partial\mu/\partial n$ in intrinsic and extrinsic graphene including exchange interaction effects, or equivalently in the HFA, which should be an excellent quantitative approximation in 2D graphene. Our calculated carrier density dependence of $\partial\mu/\partial n$ can be directly compared to experimental measurements in extrinsic graphene.

The exchange self energy is given by [2]

$$\Sigma_{x,s}(k) = - \sum_{s'} \int \frac{dq}{(2\pi)^2} n_F(\xi_{\mathbf{k}-\mathbf{q},s'}) V_c(\mathbf{q}) F_{ss'}(\mathbf{k}, \mathbf{k}-\mathbf{q}), \quad (1)$$

where $s, s' = \pm 1$ are the band indices, and since we assume $T = 0$, the fermi function $n_F(\xi) = 1$ or 0 for $\xi \equiv \epsilon - \mu$ less than or greater than 0 , respectively. $V_c(\mathbf{q}) = 2\pi e^2/(\kappa q)$ is the bare coulomb potential (κ is the background dielectric constant in the graphene layer). $F_{ss'}(\mathbf{k}, \mathbf{k}') = (1 + ss' \cos \theta_{\mathbf{k}\mathbf{k}'})/2$ arises from the wave function overlap factor, where $\theta_{\mathbf{k}\mathbf{k}'}$ is the angle between \mathbf{k} and \mathbf{k}' . We assume that the valence band is cut off at the wave vector k_c with respect to the Dirac point. The ultraviolet regularization associated with the wave vector cutoff at k_c happens at a very large wave vector, comparable to the lattice wave vector: $k_c \sim 1 \text{ \AA}^{-1}$. Within the HFA, $\mu = \epsilon_{k_F,s}^{(0)} + \Sigma_{x,s}(k_F)$, where $k_F = (4\pi|n|/g)^{1/2}$ is the Fermi wave vector.

We separate the exchange self-energy into contributions from the intrinsic electrons, Σ_x^{int} , and the extrinsic carriers, Σ_x^{ext} . That is, $\Sigma_{x,s}(k) = \Sigma_{x,s}^{\text{int}}(k) + \Sigma_{x,s}^{\text{ext}}(k)$, where

$$\Sigma_{x,s}^{\text{int}}(k) = - \int \frac{dq}{(2\pi)^2} V_c(\mathbf{q}) F_{s,-}(\mathbf{k}, \mathbf{k}-\mathbf{q}); \quad (2a)$$

$$\Sigma_{x,s}^{\text{ext}}(k) = - \sum_{s'} \int \frac{dq}{(2\pi)^2} \delta n_F(\xi_{\mathbf{k}-\mathbf{q},s'}) V_c(\mathbf{q}) F_{ss'}(\mathbf{k}, \mathbf{k}-\mathbf{q}), \quad (2b)$$

where $\delta n_F(\xi_{\mathbf{k}-\mathbf{q},s'}) = n_F(\xi_{\mathbf{k}-\mathbf{q},s'}) - \frac{1}{2}(1 - s')$ is the difference in the electron occupation from the intrinsic $T = 0$ case. Evaluating the integrals, we obtain

$$\Sigma_{x,s}^{\text{int}}(k) = \frac{e^2 k_c}{\pi \kappa} \left[-f\left(\frac{k}{k_c}\right) + sh\left(\frac{k}{k_c}\right) \right], \quad (3a)$$

$$\Sigma_{x,s}^{\text{ext}}(k) = \frac{e^2 k_F}{\pi \kappa} \left[\mp f\left(\frac{k}{k_F}\right) - sh\left(\frac{k}{k_F}\right) \right], \quad (3b)$$

where \mp in Eq. (3b) is for $\mu \gtrless 0$,

$$f(x) = \begin{cases} E(x) & \text{if } x \leq 1; \\ xE(\frac{1}{x}) - (x - \frac{1}{x})K(\frac{1}{x}) & \text{if } x > 1, \end{cases} \quad (4)$$

and

$$h(x) = \begin{cases} x[\frac{\pi}{4} \log(\frac{4}{x}) - \frac{\pi}{8}] - x \int_0^x dy y^{-3} [K(y) - E(y) - \frac{\pi}{4} y^2], & \text{for } x \leq 1; \\ x \int_0^{x^{-1}} dy [K(y) - E(y)], & \text{for } x > 1. \end{cases} \quad (5)$$

Here, $K(x)$ and $E(x)$ are the complete elliptic integral of the first and second kinds, respectively [3]. Note that the $T = 0$ exchange self-energy for a regular parabolic-band two-dimensional electron gas (2DEG) is [4] $\Sigma_x^{\text{pb}}(k) = -\frac{2e^2 k_F}{\pi \kappa} f(\frac{k}{k_F})$. At $T = 0$, $\Sigma_x(k)$ does not depend on the band-structure away from the Fermi surface (since n_F is either 1

or 0 for $\xi < 0$ and $\xi > 0$ respectively, independent of the details of the band structure) and therefore the *only* difference between $\Sigma_x(k)$ for the parabolic-band case and the intraband contribution for graphene is the difference in the wave function overlap factor $F_{ss'}(\mathbf{k}, \mathbf{k}')$. This accounts for the presence of the $f(k/k_F)$ in both the $T = 0$ expressions for $\Sigma_x^{\text{pb}}(k)$ and $\Sigma_x^{\text{ext}}(k)$.

Since we are interested in states around the Dirac point, the argument of the functions f and h in Eq. (3a), $k/k_c \ll 1$. For small x , $f(x) = E(x) \approx \frac{\pi}{2} - O(x^2)$, and $h(x) \approx x(\frac{\pi}{4} \log(\frac{4}{x}) - \frac{\pi}{8}) + O(x^3)$ [in Eq. (5), the integrand $y^{-3}[K(y) - E(y) - \frac{\pi}{4} y^2] \sim y$ as $y \rightarrow 0$, and therefore the integral $\sim x^2$ for small x]. Therefore,

$$\Sigma_{x,s}^{\text{int}}(k) = \frac{e^2}{\kappa} \left\{ -\frac{k_c}{2} + s \frac{k}{4} \left[\log\left(\frac{4k_c}{k}\right) - \frac{1}{2} + O\left(\frac{k}{k_c}\right) \right] \right\}. \quad (6)$$

The term $-e^2 k_c/(2\kappa)$ in Eq. (6) simply shifts energy zero and can be ignored. The other terms renormalize the quasiparticle velocity. Ignoring terms of order k/k_c the renormalized quasiparticle velocity is [5,6]

$$v_{\text{int}}(k) = \frac{\partial[\epsilon_{k,s}^{(0)} + \Sigma_{x,s}^{\text{int}}(k)]}{\hbar \partial k} = v_0 \left[1 + \frac{r_s^{(0)}}{4} \log\left(\frac{\tilde{k}_c}{k}\right) \right], \quad (7)$$

where $\tilde{k}_c \equiv 4e^{-3/2} k_c \approx 0.9 k_c$ and $r_s^{(0)} = e^2/(\hbar \kappa v_0)$. Experimental measurements of the quasiparticle velocity in intrinsic graphene will yield v_{int} (in the absence of phonon coupling), and not the bare velocity v_0 , which applies only for the unrealistic situation of a completely empty valence-band. This situation is analogous to the quantum electrodynamics calculation of the self-energy of a bare electron. The bare electron charge and mass of the theory are never observed. Instead, experimentally one sees the scale-dependent renormalized charge and mass, which include effects of the electron self-energy. The logarithmic dependence of the intrinsic graphene velocity is probably difficult to observe because of the smallness of the prefactor $r_s^{(0)}/4 \approx 0.2$ for graphene mounted on a SiO_2 substrate with one side exposed to air (hence, the effective κ in the graphene layer is the average of the κ of air and SiO_2 , ≈ 2.5). To see clearly the logarithmic dependence in Eq. (7), k must be varied over a fairly wide range. Furthermore, the logarithmic divergence in $v_{\text{int}}(k)$ at $k \rightarrow 0$ occurs *only* in the intrinsic graphene, and *not* in the extrinsic case.

In extrinsic graphene, $k_F \neq 0$. For $k/k_F \ll 1$, the small x expansions for $f(x)$ and $h(x)$ in Eq. (3b) yield

$$\Sigma_{x,s}^{\text{ext}}(k) = \frac{e^2}{\kappa} \left\{ \mp \frac{k_F}{2} - s \frac{k}{4} \left[\log\left(\frac{4k_F}{k}\right) - \frac{1}{2} + O\left(\frac{k}{k_F}\right) \right] \right\}, \quad (8)$$

(where \mp is for $\mu \gtrless 0$). The $\log(k)$ term in $\Sigma_{x,s}^{\text{ext}}(k \rightarrow 0)$ cancels the equivalent term in $\Sigma_{x,s}^{\text{int}}(k \rightarrow 0)$, so the sum, $\Sigma_{x,s}(k) = \Sigma_{x,s}^{\text{ext}}(k) + \Sigma_{x,s}^{\text{int}}(k)$, has a finite derivative at $k = 0$, and the renormalized velocity in the extrinsic case,

$v_{\text{ext}}(k=0) = v_0[1 + \frac{r_s^{(0)}}{4} \log(\frac{k_c}{k_F})]$, has no $k \rightarrow 0$ logarithmic divergence.

Figure 1 shows $\Sigma_x^{\text{ext}}(k)$ for graphene with $\mu > 0$ and, for comparison, $\Sigma_x^{\text{pb}}(k)$. At $k = k_F$ (and $\mu > 0$),

$$\Sigma_{x,s}^{\text{ext}}(k_F) = \frac{1}{2} \left[1 + s \left(C - \frac{1}{2} \right) \right] \Sigma_x^{\text{pb}}(k_F), \quad (9)$$

where $C \approx 0.916$ is Catalan's constant, and $\Sigma_x^{\text{pb}}(k_F) = -\pi e^2 k_F / (\pi \kappa)$. As in the case of $\Sigma_x^{\text{pb}}(k)$, the slope of $\Sigma_{x+}^{\text{ext}}(k)$ for graphene with $\mu > 0$ has a logarithmic divergence as $k \rightarrow k_F$ from both the $f(x)$ and $h(x)$ terms in Eq. (3b). We expect that the logarithmic divergence in $d\Sigma_{x+}^{\text{ext}}/dk$ will disappear when correlation effects are included, as in the case of the parabolic-band Σ_x^{pb} . Note that this logarithmic divergence has no singular pathological effect on $\partial\mu/\partial n$, the quantity of interest in this work, and is irrelevant for our purpose. The Σ_{x-}^{ext} has a finite derivative at $k = k_F$, because for an electron-doped sample there is no Fermi surface at $k = k_F$ in the valence band.

We now discuss the effect of the exchange self-energy on $\partial\mu/\partial n$. In a regular parabolic-band 2DEG with mass m , $(\frac{\partial\mu}{\partial n})_0^{\text{pb}} = \frac{2\hbar^2\pi}{m g_s g_v}$ is density independent. When interactions are included this picture changes due to exchange and correlation effects of the Coulomb potential. Within the HFA, which neglects correlation effects, $(\frac{\partial\mu}{\partial n})^{\text{pb}} = (\frac{\partial\mu}{\partial n})_0^{\text{pb}} \times (1 - \frac{\sqrt{2}}{\pi} r_s^{\text{pb}})$, where $r_s^{\text{pb}} = \sqrt{2} e^2 m / (\kappa k_F) \propto n^{-1/2}$. Thus, $\partial\mu/\partial n$ becomes negative at low enough densities. Measurements of $\partial\mu/\partial n$ in two-dimensional electron and hole gases have confirmed this behavior [7,8]. The observed change of sign in $\partial\mu/\partial n$ comes mainly from the exchange contribution to the total energy. It is known that the correlation corrections to $\partial\mu/\partial n$ beyond the HFA is not very large (<20%) [9], even at the reasonably large

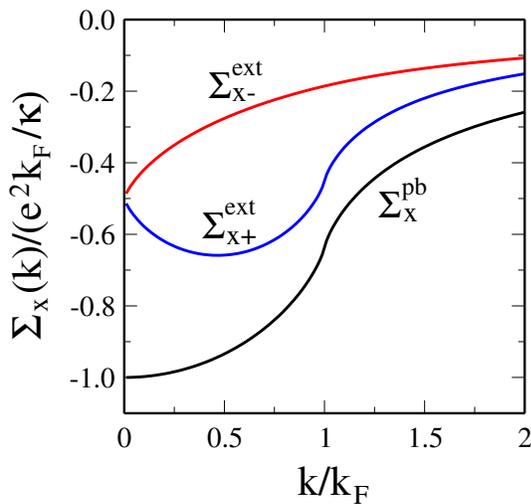


FIG. 1 (color online). Exchange self-energies for graphene (with $\mu > 0$), and for a parabolic-band 2DEG, as functions of wave vector. Note that $\Sigma_{x+}^{\text{ext}}(k) + \Sigma_{x-}^{\text{ext}}(k) = \Sigma_x^{\text{pb}}(k)$.

effective r_s ($\gg 1$) values at which the 2D semiconductor experiments have typically been carried out.

What is the contribution of exchange on $\partial\mu/\partial n$ in graphene? Using $k_F = \sqrt{\pi|n|}$, together with $\mu_s(k_F) = \epsilon_{k_F,s}^{(0)} + \Sigma_{x,s}^{\text{int}}(k_F) + \Sigma_{x,s}^{\text{ext}}(k_F)$, and Eq. (9) gives

$$\begin{aligned} \left(\frac{\partial\mu}{\partial n} \right)_{\text{ext}} &= \frac{\sqrt{\pi}}{2\sqrt{|n|}} \left\{ \hbar v_0 + \frac{e^2}{\kappa} \left[\frac{1}{4} \log\left(\frac{\tilde{k}_c}{k_F}\right) - \frac{C + \frac{1}{2}}{\pi} \right] \right\} \\ &= \left(\frac{\partial\mu}{\partial n} \right)_{\text{int}} \left[1 - \frac{C + \frac{1}{2}}{\pi} r_s^{\text{int}} \right], \end{aligned} \quad (10)$$

where $r_s^{\text{int}} = e^2 / (\kappa \hbar v_F^{\text{int}})$ [here, $v_F^{\text{int}} \equiv v_{\text{int}}(k_F)$] and $(\partial\mu/\partial n)_{\text{int}} = (\hbar v_F^{\text{int}} \sqrt{\pi}) / (2\sqrt{|n|})$ is the inverse of the density of states for intrinsic graphene. This shows that in extrinsic graphene, the exchange effect changes $\partial\mu/\partial n$ from the bare value $(\partial\mu/\partial n)_0$ by a factor of $r_s^{(0)} \left[\frac{1}{4} \times \log\left(\frac{\tilde{k}_c}{k_F}\right) - \pi^{-1}(C + \frac{1}{2}) \right]$, or from the intrinsic value $(\partial\mu/\partial n)_{\text{int}}$ by a factor of $-\pi^{-1}(C + \frac{1}{2}) r_s^{\text{int}}$. Thus, $(\partial\mu/\partial n)_{\text{ext}}$ is *enhanced* over $(\partial\mu/\partial n)_0$ by $\approx (0.25)r_s^{(0)}$ (for $n = 10^{12} \text{ cm}^{-2}$), but *reduced* from $(\partial\mu/\partial n)_{\text{int}}$ by a factor of $\approx (-0.45)r_s^{\text{int}}$. If we take single-particle band-structure graphene velocity $v_0 \approx 10^8 \text{ cm/s}$, we get $r_s^{(0)} \approx 0.9$ for SiO_2 mounted graphene, giving an exchange enhancement of approximately 20% over the bare $(\partial\mu/\partial n)_0$; see Fig. 2(a). Estimating the change with respect to $(\partial\mu/\partial n)_{\text{int}}$ is a little trickier because r_s^{int} , which depends on the intrinsic graphene velocity v_F^{int} , is unknown since the intrinsic graphene velocity is at present unknown. An approximate way to estimate the intrinsic r_s^{int} is to change $\kappa \rightarrow \kappa \kappa^*$ where κ^* is the effect of the background screening by the filled valence band [10], with $\kappa^* = 1 + \frac{\pi}{8} g_s g_v r_s^{(0)} \approx 2$, which gives $r_s^{\text{int}} \approx r_s^{(0)}/2$, leading to around a 20% decrease of $(\partial\mu/\partial n)_{\text{ext}}$ with respect to $(\partial\mu/\partial n)_{\text{int}}$. The extrinsic $\partial\mu/\partial n$ depends on κ , as shown in Fig. 2(b), but unlike the parabolic-band case, for experimentally relevant parameters $(\partial\mu/\partial n)_{\text{ext}}$ does not change sign. By using freely suspended graphene (i.e., $\kappa = 1$), the many-body renormalization can be enhanced to around 50%.

In the literature, $\partial\mu/\partial n$ is often associated with the term ‘‘compressibility,’’ defined as $K \equiv -V^{-1}(\partial V/\partial P)_{T,N}$, where N is the particle number, V is the system volume/area, $P \equiv -(\partial \mathcal{F}/\partial V)_{T,N}$ is the pressure, and \mathcal{F} is the Helmholtz free energy. It can be shown that [11] $K^{-1} = n^2(\partial\mu/\partial n)$, where $n = N/V$. In experimental papers on the compressibility of electron gases, the quantity that is measured is not actually the compressibility (after all, experimentalists do not physically compress the electron gas and measure the change in pressure) but $\partial\mu/\partial n$, which is then converted to K^{-1} by multiplication of n^2 . In graphene, it is in fact ambiguous which n should be used— (a) the free carrier density or (b) the density of the electrons in the band (i.e., free carrier density plus n_v). The answer depends on which hypothetical compressibility is being

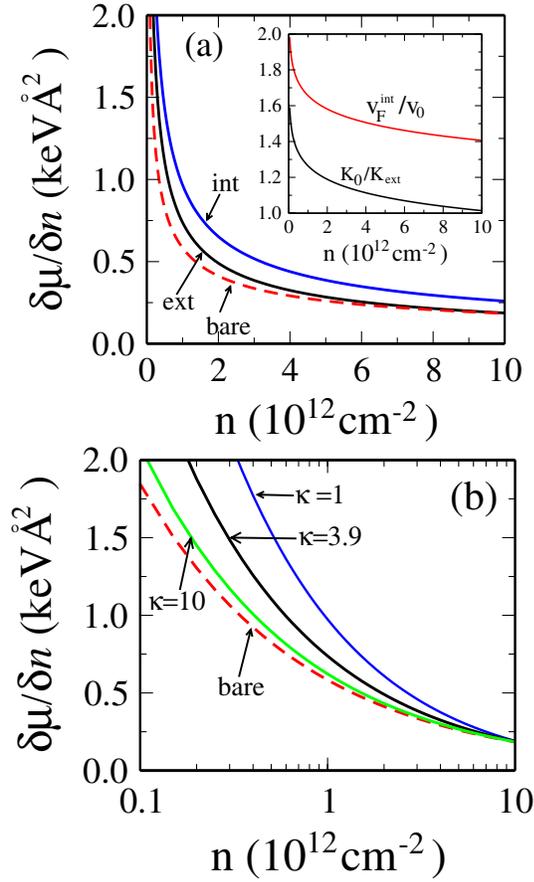


FIG. 2 (color online). (a) Calculated $\partial\mu/\partial n$ as a function of free carrier density, using the following parameters: $k_c = 1/a$ ($a = 2.46 \text{ \AA}$), $\kappa = 2.5$, $v_0 = 10^8 \text{ cm/s}$, and $r_s^{(0)} = 0.9$. The bare curve is $\partial\mu/\partial n$ of a noninteracting graphene, and the “int” and “ext” curves are for the intrinsic and extrinsic cases, respectively. The inset shows the ratios of the renormalized intrinsic velocity at k_F and the inverse of the extrinsic compressibility to their corresponding bare quantities. (b) $(\partial\mu/\partial n)_{\text{ext}}$ for different values of κ (hence, different r_s).

considered—(a) corresponds to one in which the area enclosing the free carriers is changed but the underlying lattice is kept constant, and (b) to one where the volume of the underlying lattice (and hence n_v) also changes. To avoid any ambiguities, we use the quantity $\partial\mu/\partial n$.

Before concluding, we point out that $(\partial\mu/\partial n)_{\text{ext}}$ does *not* provide a measurement of many-body quasiparticle Fermi velocity renormalization, which is given by the F_1^s parameter in Fermi liquid theory through Galilean invariance, $v_F/v_F^{\text{ren}} = 1 + F_1^s$. The renormalization of $\partial\mu/\partial n$ is related to the Fermi liquid parameters F_0^s and F_1^s through the identity, $(\frac{\partial\mu}{\partial n})^{\text{ren}} = (\frac{\partial\mu}{\partial n})(1 + F_0^s)\frac{v_F^{\text{ren}}}{v_F} = (\frac{\partial\mu}{\partial n})(\frac{1 + F_0^s}{1 + F_1^s})$. Hence, although $(\partial\mu/\partial n)_0$ for the bare system is proportional to the bare particle velocity v_0 at the Fermi surface,

$(\partial\mu/\partial n)_{\text{ext}}$ for the extrinsic case is *not* proportional to the quasiparticle velocity because of the presence of the additional Fermi liquid parameter F_0^s .

We conclude with a discussion of the possible effects of disorder and correlation on graphene $\partial\mu/\partial n$. We believe that correlation, neglected in our Hartree-Fock theory, would introduce only small quantitative corrections to our calculated results, particularly because of the relatively small values of r_s (<1) in graphene. Thus, our Fock exchange approximation for graphene $\partial\mu/\partial n$ should quantitatively be an excellent approximation. Disorder would also introduce only small quantitative corrections except at low extrinsic carrier densities ($|n| \lesssim 5 \times 10^{11} \text{ cm}^{-2}$) associated with the so-called “minimal graphene conductivity” regime, where random charged impurities in the substrate introduce [12,13] inhomogeneous electron-hole puddles in the graphene layer which would lead to random spatial variations in $\partial\mu/\partial n$ over 5–20 nm (10–100 meV) length (energy) scales. Finally, finite temperature would have little effect on our results because $E_F > 1000 \text{ K}$ in the usual density range of experimental interest ($\geq 5 \times 10^{11} \text{ cm}^{-2}$).

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Note added.—After submission of this manuscript, we received a paper [14] reporting an experimental observation of the exchange contribution to $\partial\mu/\partial n$ that is consistent with our theory.

- [1] For a review see A. K. Geim and K. S. Novoselov, *Nat. Mater.* **6**, 183 (2007), and references therein.
- [2] See, e.g., G. D. Mahan, *Many Particle Physics* (Kluwer/Plenum, New York, 2000), 3rd ed.
- [3] See, e.g., I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, Oxford, 2007), 7th ed.
- [4] A. V. Chaplik, *Zh. Eksp. Teor. Fiz.* **60**, 1845 (1971) [*Sov. Phys. JETP* **33**, 997 (1971)]; F. Stern, *Phys. Rev. Lett.* **30**, 278 (1973).
- [5] J. González *et al.*, *Phys. Rev. B* **59**, R2474 (1999); S. Das Sarma *et al.*, *Phys. Rev. B* **75**, 121406(R) (2007).
- [6] Y. Barlas *et al.*, *Phys. Rev. Lett.* **98**, 236601 (2007).
- [7] J. P. Eisenstein *et al.*, *Phys. Rev. Lett.* **68**, 674 (1992).
- [8] S. Shapira *et al.*, *Phys. Rev. Lett.* **77**, 3181 (1996).
- [9] B. Tanatar and D. M. Ceperley, *Phys. Rev. B* **39**, 5005 (1989).
- [10] E. H. Hwang and S. Das Sarma, *Phys. Rev. B* **75**, 205418 (2007).
- [11] See, e.g., Franz Schwabl, *Statistical Mechanics* (Springer, Berlin, 2006), 2nd ed., p. 85.
- [12] E. H. Hwang *et al.*, *Phys. Rev. Lett.* **98**, 186806 (2007).
- [13] V. M. Galitski *et al.*, arXiv:cond-mat/0702117 [*Phys. Rev. B* (to be published)].
- [14] J. Martin *et al.*, arXiv:condmat/0705.2180.