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Relativistic momentum and kinetic energy, and $E = mc^2$

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Abstract
Based on relativistic velocity addition and the conservation of momentum and energy, I present simple derivations of the expressions for the relativistic momentum and kinetic energy of a particle, and for the formula $E = mc^2$.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The standard formal way that upper-level introductory undergraduate textbooks [1] obtain the expressions for the relativistic momentum **p**, the relativistic kinetic energy T and the mass–energy relationship $E = mc^2$ is by first introducing Lorentz transformations and 4-vectors, and then defining the 4-momentum vector $p^\mu = m\,\mathrm{d}x^\mu/\mathrm{d}\tau$ $(\mu = 0, 1, 2, 3)$, where τ is the proper time. The temporal and spatial components of p^μ reduce in the non-relativistic limit to $mc^2 + \frac{1}{2}mv^2$ and $m\mathbf{v}$, respectively, and are therefore postulated to be the relativistic generalizations of the total energy and momentum. It is then asserted that in an isolated system, all the components of p^μ are conserved. In this approach, the student is first confronted with an unfamiliar quantity, the 4-vector, followed by an assertion of the conservation of its components which must be taken 'on faith', since there is no way *a priori* to justify it. This approach is pedagogically rather unsatisfactory. As a result, there have been many papers describing derivations of the relativistic expressions that are based on more physical grounds [2].

This paper describes new simple and concise derivations of the relativistic forms of **p** and T, and $E = mc^2$, all based on (i) the conservation of momentum and energy in the collisions of two particles and (ii) the velocity addition rules. Momentum and energy conservation should be concepts familiar to students, and the velocity addition rules can be quite simply derived from the constancy of the speed of light in all inertial reference frames [3].

To simplify the algebra, velocities in this paper are expressed in units of c, the speed of light. Hence velocities are dimensionless, and $c = 1$. To obtain the standard dimensional expressions, replace all velocities in the expressions given here by $v \to v/c$ and multiply all masses by c^2 in order to obtain energy. Also, in this paper primes on variables denote 'after collision'.

2. The derivations

In all three derivations, collisions are analysed in the centre-of-momentum frame of reference, $S_{\rm cm}$, in which both particles have momenta that are equal in magnitude and opposite in direction, and in the laboratory frame of reference, $S_{\rm lab}$, in which one of the particles is initially at rest. It is self-evident that the appropriate conservation laws are obeyed in $S_{\rm cm}$. Imposition of conservation laws in $S_{\rm lab}$ gives the desired expressions.

2.1. Relativistic velocity transformations

Let us recall the relativistic velocity transformation rules. Let $\tilde{S}$ be an inertial frame moving with velocity $(u, 0)$ with respect to frame S. If a particle has velocity (v_x, v_y) in frame S, the components of its velocity in frame $\tilde{S}$ are [1, 3]

$$\tilde{v}_x = \frac{v_x - u}{1 - v_x u}, \tag{1a}$$

$$\tilde{v}_y = \frac{v_y\sqrt{1-u^2}}{1 - v_x u}. \tag{1b}$$

2.2. Relativistic momentum

From dimensional analysis and the vector[1] nature of momentum, the momentum of a particle of mass m travelling with velocity $\mathbf{v}$ must have the form

$$\mathbf{p} = m\gamma(v)\mathbf{v}, \tag{2}$$

where $\gamma(v)$ is a function of $v \equiv |\mathbf{v}|$ that is to be determined. Since $\mathbf{p} = m\mathbf{v}$ for non-relativistic velocities, $\gamma(0) = 1$.

Consider the case where the particles are identical; hence, $m_1 = m_2 = m$. Let the motion of the particles be in the x–y plane and their initial velocities in $S_{\rm cm}$ be $\pm(v, 0)$. Assume that the particles barely graze each other, so that in the collision each particle picks up a very small y-component of the velocity of magnitude δv in $S_{\rm cm}$. (See figure 1(a).) Their speeds in $S_{\rm cm}$ do not change because the collision is elastic, and hence their velocities after the collision are $\pm(\sqrt{v^2 - (\delta v)^2}, \delta v) \approx \pm(v, \delta v)$, to first order in δv. Because δv is assumed to be very small, we ignore all terms of order $(\delta v)^2$ and higher.

Now consider the collision in the laboratory frame of reference $S_{\rm lab}$ that moves with velocity $(-v, 0)$ with respect to $S_{\rm cm}$. (See figure 1(b).) The pre-collision velocities of the particles in $S_{\rm lab}$, using (1*a*) and (1*b*) on the $S_{\rm cm}$ velocities $\pm(v, 0)$, are $\mathbf{v}_{1,\rm lab} = (w, 0)$, where

$$w = \frac{2v}{1+v^2}, \tag{3}$$

and $\mathbf{v}_{2,\rm lab} = (0, 0)$. After the collision, transforming the post-collision $S_{\rm cm}$ velocities $\pm(v, \delta v)$ to the $S_{\rm lab}$ frame we obtain, to the first order in δv,

$$\mathbf{v}'_{1,\rm lab} \approx \left(w, \frac{\delta v\sqrt{1-v^2}}{1+v^2}\right), \tag{4a}$$

[1] Here, the terms 'vector' and 'scalar' are used in the non-relativistic (i.e. not the 4-vector) sense.

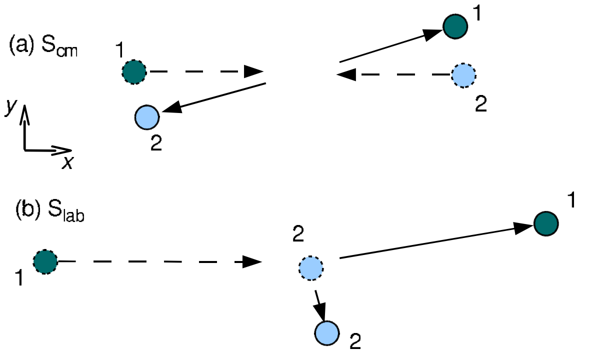

Figure 1. Grazing collision between two particles of equal mass in (a) centre of momentum and (b) laboratory frames of reference. Dashed and solid lines indicate before and after the collision, respectively.

$$\mathbf{v}'_{2,\text{lab}} \approx \left(0, -\frac{\delta v\sqrt{1-v^2}}{1-v^2}\right). \tag{4b}$$

The y-component of the total momentum before the collision is zero, and hence by conservation of momentum, after the collision

$$(\mathbf{p}'_{1,\text{lab}} + \mathbf{p}'_{2,\text{lab}})_y = m\gamma(v'_{1,\text{lab}})v'_{1,\text{lab},y} + m\gamma(v'_{2,\text{lab}})v'_{2,\text{lab},y} = 0. \tag{5}$$

To the first order in δv, (5) together with (4a) and (4b) gives

$$\left[\frac{\gamma(w)}{1+v^2} - \frac{\gamma(0)}{1-v^2}\right]\delta v = 0, \tag{6}$$

which implies that the term in the square parentheses vanishes. This together with (3) and $\gamma(0) = 1$ (the non-relativistic limit) gives the desired result,

$$\gamma(w) = \frac{1+v^2}{1-v^2} = \left(1 - \left[\frac{2v}{1+v^2}\right]^2\right)^{-1/2} = (1-w^2)^{-1/2}. \tag{7}$$

2.3. Relativistic kinetic energy

Dimensional analysis and the scalar[2] property of kinetic energy imply that its form is

$$T = mG(v), \tag{8}$$

where m is the mass of the particle, $v = |\mathbf{v}|$ is its speed and the function $G(v)$ is to be determined. Since the kinetic energy of a stationary object vanishes, $G(0) = 0$.

Consider an elastic collision between two particles of mass m and $M \gg m$, with speeds in S_{cm} of v and V respectively, in which the outgoing and incoming velocity vectors are perpendicular. (See figure 2(a).) Assume that the mass M is so large that in frame S_{cm} its speed $V \ll 1$, and hence we can use the non-relativistic expressions for the momentum and kinetic energy of mass M. The magnitudes of the momenta of m and M are equal in S_{cm}, implying

$$m\gamma(v)v = MV. \tag{9}$$

[2] See footnote 1.

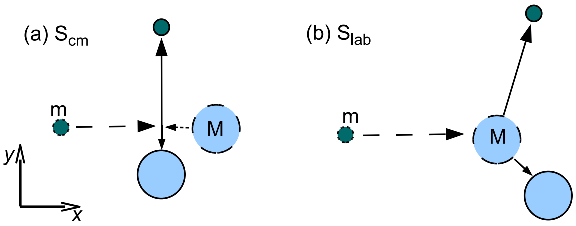

Figure 2. Collision between particles of mass m and $M \gg m$ in (a) centre of momentum and (b) laboratory frames of reference. Dashed and solid lines indicate before and after the collision, respectively.

The S_{cm} frame pre- and post-collision velocities of mass m are $\mathbf{v}_{cm} = (v, 0)$ and $\mathbf{v}'_{cm} = (0, v)$, respectively, and of mass M are $\mathbf{V}_{cm} = (-V, 0)$ and $\mathbf{V}'_{cm} = (0, -V)$, respectively. Transforming these to the S_{lab} frame which moves at velocity $(-V, 0)$ with respect to S_{cm} (see figure 2(b)) using (1a) and (1b) gives $\mathbf{v}_{lab} = ((v + V)/(1 + vV), 0)$, $\mathbf{v}'_{lab} = (V, v\sqrt{1 - V^2})$, $\mathbf{V}_{lab} = (0, 0)$ and $\mathbf{V}'_{lab} = (V, -V\sqrt{1 - V^2})$. By conservation of kinetic energy in an elastic collision in the S_{lab} frame and (8),

$$mG(v_{lab}) = mG(v'_{lab}) + \frac{M}{2}V'^2_{lab}. \tag{10}$$

Expanding v_{lab}, v'_{lab} and V'_{lab} to the first order in V gives

$$v_{lab} \approx v + V(1 - v^2), \tag{11a}$$

$$v'_{lab} \approx v, \tag{11b}$$

$$V'_{lab} \approx \sqrt{2}V. \tag{11c}$$

Substituting these into (10) and Taylor expanding the $G(v_{lab})$ term on the left-hand side about v give, to the first order in V,[3]

$$m\left(G(v) + \left[\frac{\mathrm{d}G(u)}{\mathrm{d}u}\right]_v V(1 - v^2)\right) = mG(v) + MV^2. \tag{12}$$

Substituting $MV^2 = m\gamma(v)vV$ (from (9)) into (12) leads to

$$\left[\frac{\mathrm{d}G}{\mathrm{d}u}\right]_v = \frac{\gamma(v)v}{1 - v^2} = \frac{v}{(1 - v^2)^{3/2}}, \tag{13}$$

which upon integration yields

$$G(v) - G(0) = \left[\frac{1}{(1 - u^2)^{1/2}}\right]_{u=0}^{u=v} = \gamma(v) - 1. \tag{14}$$

Since $G(0) = 0$, this yields (from (8) and reintroducing c) $T = m(\gamma(v) - 1)c^2$.

2.4. $E = mc^2$

Consider the initial situation as in section 2.3, except that the speed V of mass M can be relativistic, and after collision the two particles merge into one composite particle. In S_{cm}, $M\gamma(V)V = m\gamma(v)v$, and after the collision the composite particle is stationary. In S_{lab}

[3] The term $2MV^2$ in (12) is actually the *first* order in V, because M is of order V^{-1} (see (9)).

which moves with velocity $(-V, 0)$ with respect to S_{cm}, before the collision particle M is stationary and particle m moves with velocity $\mathbf{v}_{\text{lab}} = \left(\frac{v+V}{1+vV}, 0\right)$, and after the collision the composite particle moves with velocity $(V, 0)$.

The magnitude of the total momentum in S_{lab} before the collision is $P_{\text{lab}} = m\gamma(v_{\text{lab}})v_{\text{lab}} = m\gamma(v)\gamma(V)(v+V)$. If the mass of the composite particle does not change, then the momentum of the composite particle after the collision in S_{lab} would be $(M+m)\gamma(V)V \neq P_{\text{lab}}$ in general, violating conservation of momentum. Therefore, the mass of the composite particle must change by Δm such that momentum is conserved in S_{lab}, i.e.

$$m\gamma(v)\gamma(V)(v+V) = (M+m+\Delta m)\gamma(V)V. \tag{15}$$

Using $m\gamma(v)v = M\gamma(V)V$ to eliminate v on the left-hand side of (15) and cancelling $\gamma(V)V$ on both sides give

$$\Delta m = m(\gamma(v) - 1) + M(\gamma(V) - 1). \tag{16}$$

From section 2.3, the right-hand side of (16) is equal to $-\Delta T$, the total change in kinetic energy in S_{cm} (since particles m and M start with speeds v and V, respectively, and both are stationary at the end). By conservation of total energy, $\Delta E + \Delta T = 0$, where ΔE is the change of energy associated with the change in mass. Hence, $\Delta E = -\Delta T = \Delta m$ or (reintroducing c and making the plausible assumption that a zero mass object with zero velocity has zero energy[4]) $E = mc^2$. Finally, combining the results of sections 2.3 and 2.4 gives the total energy of a particle of mass m moving with speed v, $E + T = E_{\text{total}} = m\gamma(v)c^2$.

3. Concluding remarks

It should be noted that these derivations do not guarantee that the momentum and total energy are conserved in all inertial reference frames or in all collisions. They only show the forms that the momentum, kinetic energy and energy–mass relation must have, given momentum and energy conservation. Once these expressions are known, when the 4-momentum is introduced its components will be recognized as the total energy and momentum. The covariance of the momentum 4-vector can then be used to demonstrate the momentum and total energy conservation in all inertial frames. The conservation of momentum can be shown to be a consequence of conservation of energy[5], and, as befitting an experimental science, the conservation of energy ultimately depends on experimental observations.

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[5] See Penrose R and Rindler W, Ehlers J, Rindler W and Penrose R, Siman Y and Husson N, and Feigenbaum M J and Mermin N D in [2]. Note that these derivations explicitly or implicitly assume isotropy of space and translational invariance.

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