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Solving boundary-value electrostatics problems using Green's reciprocity theorem

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Abstract

Formal solutions to electrostatics boundary-value problems are derived using Green's reciprocity theorem. This method provides a more transparent interpretation of the solutions than the standard Green function derivation. An energy-based argument for the reciprocity theorem is also presented.

I. INTRODUCTION

In electrostatic boundary-value problems, one solves for the potential $\Phi(\mathbf{r})$ in a volume \mathcal{V} , given the charge density $\rho(\mathbf{r})$ in \mathcal{V} and boundary conditions on the surface \mathcal{S} of \mathcal{V} . The Dirichlet and Neumann problems refer respectively to the cases where the boundary conditions are given in terms of the potential Φ and the electric field $\mathbf{E} = -\nabla\Phi$. These boundary-value problems are typically covered in graduate-level texts,¹⁻³ and the formal solutions to these problems are usually derived using Green function techniques. Explicitly, these solutions are

$$\text{Dirichlet: } \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{r}') G_D(\mathbf{r}, \mathbf{r}') d^3r' - \frac{1}{4\pi} \int_{\mathcal{S}} \Phi(\mathbf{r}') \frac{\partial G_D(\mathbf{r}, \mathbf{r}')}{\partial n'} da', \quad (1a)$$

$$\text{Neumann: } \Phi(\mathbf{r}) = \langle \Phi \rangle_{\mathcal{S}} + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{r}') G_N(\mathbf{r}, \mathbf{r}') d^3r' + \frac{1}{4\pi} \int_{\mathcal{S}} \frac{\partial \Phi(\mathbf{r}')}{\partial n'} G_N(\mathbf{r}, \mathbf{r}') da', \quad (1b)$$

where $\langle \Phi \rangle_{\mathcal{S}}$ is the average of Φ on \mathcal{S} , $\partial/\partial n'$ is the derivative with respect to the normal direction of surface \mathcal{S} (pointing *out* of volume \mathcal{V}), and $G_D(\mathbf{r}, \mathbf{r}')$ and $G_N(\mathbf{r}, \mathbf{r}')$ are the Dirichlet and Neumann Green functions, respectively. Both $G_D(\mathbf{r}, \mathbf{r}')$ and $G_N(\mathbf{r}, \mathbf{r}')$ satisfy

$$\nabla'^2 G_{D,N}(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \quad (2)$$

for \mathbf{r}' in \mathcal{V} . The difference lies in their boundary conditions for \mathbf{r}' on \mathcal{S} ; *viz.*,

$$\text{Dirichlet: } G_D(\mathbf{r}, \mathbf{r}') = 0; \quad (3a)$$

$$\text{Neumann: } \frac{\partial G_N}{\partial n'}(\mathbf{r}, \mathbf{r}') = 0, \quad G_N(\mathbf{r}, \mathbf{r}') = -\frac{4\pi}{|\mathcal{S}|}, \quad (3b)$$

where $|\mathcal{S}|$ is the surface area of \mathcal{S} .

The main purpose of this note is to present an alternative derivation of Eqs. (1a) and (1b), based on Green's reciprocity theorem,^{4,5} which is less mathematically involved than the standard Green function derivation and which gives a physical interpretation to the Green function terms G_D and G_N .

II. GREEN'S RECIPROCITY THEOREM

The reciprocity theorem states that two *completely unrelated* charge distributions $\rho(\mathbf{r}')$ and $\rho^*(\mathbf{r}')$ with corresponding electrostatic potentials $\Phi(\mathbf{r}')$ and $\Phi^*(\mathbf{r}')$, respectively, satisfy the relationship

$$\int_{\text{all space}} \Phi(\mathbf{r}') \rho^*(\mathbf{r}') d^3r' = \int_{\text{all space}} \Phi^*(\mathbf{r}') \rho(\mathbf{r}') d^3r'. \quad (4)$$

Eq. (4) can be derived by evaluating $\int \nabla\Phi(\mathbf{r}') \cdot \nabla\Phi^*(\mathbf{r}') d^3r'$ two different ways using integration by parts and utilizing Poisson's equation, $-\nabla^2\Phi = \rho/\epsilon_0$.^{4,5} It can also be inferred by considering the electrostatic energy required to assemble the charge distribution consisting of *both* $\rho(\mathbf{r}')$ and $\rho^*(\mathbf{r}')$. Consider the *gedanken* experiment where the charge density $\rho(\mathbf{r}') + \rho^*(\mathbf{r}')$ is created by assembling ρ and ρ^* infinitely far apart, and then bringing them together. If we keep ρ fixed and bring in ρ^* , the work done is $\int \Phi(\mathbf{r}') \rho^*(\mathbf{r}') d^3r'$. Conversely, if we keep ρ^* fixed and move ρ towards ρ^* , the work done is $\int \Phi^*(\mathbf{r}') \rho(\mathbf{r}') d^3r'$. Since the work done in either case must be equal, we immediately obtain Eq. (4). Note also that the energy argument holds for any form of the potential (so long as it is linear), whereas the integration by parts proof relies explicitly on the potential having a $1/r$ form (through Poisson's equation).

We shall derive the formal solutions to the Dirichlet and Neumann boundary-value problems by taking ρ to be the given charge distribution and picking a ρ^* which, when applied to Eq. (4), reproduces Eqs. (1a) and (1b).

III. DIRICHLET BOUNDARY CONDITION

The Dirichlet boundary condition specifies the potential $\Phi(\mathbf{r}')$ on a boundary \mathcal{S} . For the purpose of this derivation, we shall assume that the boundary condition is caused by the given charge density $\rho(\mathbf{r}')$ in \mathcal{V} *and* possibly other charges $\bar{\rho}(\mathbf{r}')$ that lie in the region of space outside \mathcal{V} , which we denote as $\bar{\mathcal{V}}$ [see Fig. 1(a)].

We pick $\rho^*(\mathbf{r}')$ by imagining that all of $\bar{\mathcal{V}}$ is occupied by a grounded conductor and that there is a unit charge at position \mathbf{r} [see Fig. 1(b)]. Let the potential created by this configuration be $\Phi_{D,\mathbf{r}}^*(\mathbf{r}')$. (The subscript \mathbf{r} reminds us that the potential depends on the placement of the unit charge.) Since all free excess charge in a conductor moves to the surface, the unit charge at \mathbf{r} induces a charge $\sigma_{D,\mathbf{r}}^*(\mathbf{r}')$ on the surface of the conductor, which by construction coincides with the boundary \mathcal{S} . Substituting the charge distribution $\rho^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') + \int_{\mathcal{S}} \sigma_{D,\mathbf{r}}^*(\mathbf{r}'') \delta(\mathbf{r}' - \mathbf{r}'') da''$ into Eq. (4) yields

$$\Phi(\mathbf{r}) + \int_{\mathcal{S}} \sigma_{D,\mathbf{r}}^*(\mathbf{r}') \Phi(\mathbf{r}') da' = \int_{\mathcal{V}} \Phi_{D,\mathbf{r}}^*(\mathbf{r}') \rho(\mathbf{r}') d^3r' + \int_{\bar{\mathcal{V}}} \Phi_{D,\mathbf{r}}^*(\mathbf{r}') \bar{\rho}(\mathbf{r}') d^3r'. \quad (5)$$

The second term on the right-hand side of Eq. (5) vanishes because the potential in $\bar{\mathcal{V}}$ for the starred system is zero. Furthermore, since $\sigma_{D,\mathbf{r}}^*$ is on the surface of the conductor, it is related to the normal component of the electric field by Gauss' law,⁶

$$\frac{\sigma_{D,\mathbf{r}}^*(\mathbf{r}')}{\epsilon_0} = \frac{\partial \Phi_{D,\mathbf{r}}^*(\mathbf{r}')}{\partial n'}. \quad (6)$$

Using this in Eq. (5) yields the desired solution

$$\Phi(\mathbf{r}) = \int_{\mathcal{V}} \Phi_{D,\mathbf{r}}^*(\mathbf{r}') \rho(\mathbf{r}') d^3r' - \epsilon_0 \int_{\mathcal{S}} \frac{\partial \Phi_{D,\mathbf{r}}^*(\mathbf{r}')}{\partial n'} \Phi(\mathbf{r}') da'. \quad (7)$$

Comparing this equation with Eq. (1a), we can immediately make the identification

$$G_D(\mathbf{r}, \mathbf{r}') = 4\pi\epsilon_0 \Phi_{D,\mathbf{r}}^*(\mathbf{r}'). \quad (8)$$

Thus, as expected, the Dirichlet Green function $G_D(\mathbf{r}, \mathbf{r}')$ is proportional to the potential at \mathbf{r}' due to a point charge at \mathbf{r} , given that $\bar{\mathcal{V}}$ is filled with a grounded conductor, or equivalently, that the surface \mathcal{S} is grounded (by the uniqueness theorem⁷).

IV. NEUMANN BOUNDARY CONDITION

In Neumann boundary-value problems, the perpendicular component of the electric field $\partial\Phi/\partial n'$ is given on \mathcal{S} . We assume again that the electric field at the surface \mathcal{S} is due to a combination of the charges $\rho(\mathbf{r}')$ in \mathcal{V} and possibly additional charges $\bar{\rho}(\mathbf{r}')$ in $\bar{\mathcal{V}}$. For $\rho_{N,\mathbf{r}}^*(\mathbf{r}')$,

we choose a unit charge at \mathbf{r} in \mathcal{V} and additional (to be specified later) charges in $\bar{\mathcal{V}}$. Using these charge distributions in Eq. (4) gives

$$\Phi(\mathbf{r}) + \int_{\bar{\mathcal{V}}} \Phi(\mathbf{r}') \rho_{N,\mathbf{r}}^*(\mathbf{r}') d^3r' = \int_{\mathcal{V}} \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \rho(\mathbf{r}') d^3r' + \int_{\bar{\mathcal{V}}} \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \bar{\rho}(\mathbf{r}') d^3r'. \quad (9)$$

Moving the second term on the left hand side over to the right and using Poisson's equation yields

$$\Phi(\mathbf{r}) = \int_{\mathcal{V}} \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \rho(\mathbf{r}') d^3r' + \epsilon_0 \int_{\bar{\mathcal{V}}} [\Phi(\mathbf{r}') \nabla'^2 \Phi_{N,\mathbf{r}}^*(\mathbf{r}') - \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \nabla'^2 \Phi(\mathbf{r}')] d^3r'. \quad (10)$$

We can transform this equation with the help of Green's theorem,⁸ which states $\int_{\bar{\mathcal{V}}} (\phi \nabla'^2 \psi - \psi \nabla'^2 \phi) d^3r' = \int_{\bar{\mathcal{S}}} (\phi \hat{\mathbf{n}} \cdot \nabla' \psi - \psi \hat{\mathbf{n}} \cdot \nabla' \phi) da'$, into⁹

$$\Phi(\mathbf{r}) = \int_{\mathcal{V}} \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \rho(\mathbf{r}') d^3r' + \epsilon_0 \int_{\bar{\mathcal{S}}} \left[-\Phi(\mathbf{r}') \frac{\partial \Phi_{N,\mathbf{r}}^*}{\partial n'}(\mathbf{r}') + \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \frac{\partial \Phi}{\partial n'}(\mathbf{r}') \right] da'. \quad (11)$$

Ideally, we would like to choose a ρ^* which gives $\partial \Phi_{N,\mathbf{r}}^*/\partial n' = 0$ over the entire boundary $\bar{\mathcal{S}}$, so that the second term on the right hand side of Eq. (11) vanishes. However, this is impossible because there is a unit charge in \mathcal{V} , and hence Gauss' law states that $-\int_{\bar{\mathcal{S}}} \partial \Phi_{N,\mathbf{r}}^*/\partial n' da' = \epsilon_0^{-1}$. The simplest choice consistent with Gauss' law is to assume ρ^* creates a constant $-\partial \Phi_{N,\mathbf{r}}^*/\partial n' = (\epsilon_0 |S|)^{-1}$ on $\bar{\mathcal{S}}$. The existence of such a ρ^* is shown in the Appendix. Using this choice, Eq. (11) becomes

$$\Phi(\mathbf{r}) = \int_{\mathcal{V}} \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \rho(\mathbf{r}') d^3r' + \langle \Phi \rangle_{\bar{\mathcal{S}}} + \frac{1}{\epsilon_0} \int_{\bar{\mathcal{S}}} \Phi_{N,\mathbf{r}}^*(\mathbf{r}') \frac{\partial \Phi}{\partial n'}(\mathbf{r}') da'. \quad (12)$$

As in the Dirichlet case, comparison of Eqs. (1b) and (12) allow us to identify $G_N(\mathbf{r}, \mathbf{r}') = 4\pi\epsilon_0 \Phi_{N,\mathbf{r}}^*(\mathbf{r}')$.

V. CONCLUSION

The formal solution to the Dirichlet boundary-value problem can be derived using material familiar to beginning junior/senior-level electromagnetic theory students. For example, solving problems with point charges and grounded conductors using the method of images is typically introduced early in junior/senior-level electromagnetic theory courses. Thus in

principle students can also be introduced to simple Dirichlet boundary-value problems fairly early in these courses. As the derivation of the solution to the Neumann problem requires a little more mathematical sophistication than the solution to the Dirichlet problem, I would expect that discussion of Neumann problems will be deferred to a more advanced course.

APPENDIX: PROOF OF EXISTENCE OF $\rho^*(\mathbf{r}')$ WHICH GIVES

$$-\partial\Phi_{N,r}^*/\partial n' = (\epsilon_0 |S|)^{-1} \text{ ON } \mathcal{S}$$

Let us assume that $\rho^*(\mathbf{r}')$ is given by a unit charge at \mathbf{r} and a surface charge density $\sigma_N(\mathbf{r}')$ on the surface \mathcal{S} ; *i.e.*, $\rho^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') + \int_{\mathcal{S}} \sigma_N^*(\mathbf{r}'') \delta(\mathbf{r}' - \mathbf{r}'') da''$. Let $E_{\perp,r}(\mathbf{r}')$ be the perpendicular component of the electric field just inside \mathcal{S} due to the unit charge at \mathbf{r} . To prove the assertion, we show that there is a $\sigma_N^*(\mathbf{r}')$ which produces a perpendicular component of the electric field $\delta E_{\perp}(\mathbf{r}') = (\epsilon_0 |S|)^{-1} - E_{\perp,r}(\mathbf{r}')$ just inside \mathcal{S} .

If we consider \mathbf{r}' on \mathcal{S} to be the index of an infinite-dimensional vector, then we can formally write the relationship between $\sigma_N^*(\mathbf{r}')$ and $\delta E_{\perp}(\mathbf{r}')$ as $\vec{L}\vec{\sigma}_N^* = \delta\vec{E}_{\perp}$, where \vec{L} is a linear operator. To be able write the inverted equation $\vec{\sigma}_N^* = \vec{L}^{-1}\delta\vec{E}_{\perp}$, we must show that \vec{L}^{-1} exists in the subspace spanned by $\delta\vec{E}_{\perp}$.

Since Gauss' law implies $\int_{\mathcal{S}} \delta E_{\perp}(\mathbf{r}') da' = 0$, we need only consider the subspace of functions whose integral over the surface \mathcal{S} vanishes; *i.e.*, $\int_{\mathcal{S}} \sigma_N(\mathbf{r}') da' = 0$. The matrix \vec{L} is invertible if none of its eigenvalues in this subspace is zero. We see below that a $\lambda_{\alpha} = 0$ in this subspace yields a contradiction.

Physically, $\lambda_{\alpha} = 0$ indicates that there is a non-zero charge density on \mathcal{S} which gives $E_{\perp} = 0$ just inside \mathcal{S} . From the uniqueness theorem,⁷ this implies the electric field \mathbf{E} vanishes in all of \mathcal{V} . Consider the path integral $\oint \mathbf{E} \cdot d\mathbf{l}$ which starts a point on \mathcal{S} which has net positive charge, follows the field line inside $\bar{\mathcal{V}}$ until it ends up at a negative charge on \mathcal{S} (there must be regions of positive and negative charge since $\int_{\mathcal{S}} \sigma_{\alpha}(\mathbf{r}') da' = 0$) and then continues inside \mathcal{V} back to its starting point. The line integral outside \mathcal{V} is positive and the line integral inside \mathcal{V} is zero, implying $\oint \mathbf{E} \cdot d\mathbf{l} \neq 0$ which violates the laws of electrostatics.¹⁰

REFERENCES

- ¹ John David Jackson, *Classical Electrodynamics* (Wiley, New York, 1999), pp. 35–40.
- ² Walter Greiner, *Classical Electrodynamics* (Springer, Berlin, 1998), pp. 48–50.
- ³ Richard S. Elliot, *Electromagnetics: History and Applications* (IEEE Press, New York, 1993), pp. 172–175.
- ⁴ David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewoods Cliffs, NJ, 1999), pp. 157–158.
- ⁵ Reference 1, p. 52; reference 3, p. 191.
- ⁶ The usual minus sign in relationship between the field and the derivative of the potential is absent because $\partial/\partial n'$ is defined to be positive pointing out of \mathcal{V} into $\overline{\mathcal{V}}$.
- ⁷ See *e.g.*, reference 1, pp. 37–38; reference 4 pp. 116–120.
- ⁸ See *e.g.*, reference 1, pp. 35–37; reference 4 p. 56.
- ⁹ Note that a unit vector normal to a point on the surface $\overline{\mathcal{S}}$ is the negative of the unit vector normal to \mathcal{S} . This accounts for change of sign of the terms in the square parentheses in Eq. (11).
- ¹⁰ This argument assumes \mathcal{S} is a connected surface. If the surface is composed of distinct disconnected sections \mathcal{S}_i , one might argue that the net positive and negative charges reside on different sections, and it is impossible to find a path through $\overline{\mathcal{V}}$ from one section to the other; *e.g.*, the case where \mathcal{V} is the volume bounded by spherical surfaces $r_1 < r < r_2$, and \mathcal{S}_1 (\mathcal{S}_2) is the spherical surface of radius r_1 (r_2). In this case, however, since $\mathbf{E} = 0$ in \mathcal{V} , all the field lines would have to start and terminate on the same surface, which is impossible if there is only purely positive or negative charge on that surface. Finally, one might argue that there are situations where it is impossible find a path through \mathcal{V} from the disconnected surfaces; *e.g.*, the case where \mathcal{V} is given by $r < r_1$ and $r > r_2$

(where $r_1 < r_2$). In this case, the entire argument is moot because \mathcal{V} consists of distinct disconnected volumes, and each disconnected volume can each be treated as a separate boundary-value problem without regard to the others.

Figures

Figure 1. (a) The Dirichlet boundary-value problem. The charge $\rho(\mathbf{r}')$ is specified in volume \mathcal{V} , and the potential is specified on the surface \mathcal{S} . It is assumed that the potential on \mathcal{S} is caused by $\rho(\mathbf{r}')$ in \mathcal{V} and $\bar{\rho}(\mathbf{r}')$ in $\bar{\mathcal{V}}$. (b) The starred system. There is a unit charge at \mathbf{r} in \mathcal{V} , and $\bar{\mathcal{V}}$ is filled with a grounded conductor. The unit charge at \mathbf{r} induces a surface charge $\sigma_{D,\mathbf{r}}^*$ on \mathcal{S} .



