Non-linear forced harmonic motion – the damped and driven pendulum

Restoring force: \( F_s = -mg \sin \theta \)

Damping force: \( F_d = -c \dot{\theta} = -cL \omega \)

Driving force: \( F_f = F_0 \sin \omega_f t \)

Net force: \( F_{\text{net}} = -mg \sin \theta - L \omega + F_0 \sin \omega_f t \)

Newton’s 2nd law: \( mL \frac{d^2 \theta}{dt^2} = -mg \sin \theta - cL \frac{d\theta}{dt} + F_0 \sin \omega_f t \)

Define: \( \frac{g}{L} = \Omega^2 \quad \frac{cL}{m} = q \)

Equation of motion, 2nd order differential equation:

\[
\dot{\theta} + q \dot{\theta} + \Omega^2 \sin \theta = \frac{F_0}{m} \sin \omega_f t
\]

Important (angular) frequencies:

\( \omega = \frac{d\theta}{dt} = \dot{\theta} \) Angular velocity of the pendulum

\( \omega_f \) Driving frequency, driving period \( T_f = \frac{2\pi}{\omega_f} \)

\( \Omega = \sqrt{\frac{g}{L}} \) Natural frequency of the undamped oscillator

Motion of the damped and driven pendulum for small driving amplitude \( F_0/m \).

Damped and driven pendulum with \( F_0/m = 0.2, \omega_f = 0.67, q = 0.5 \)

After the transient time, the motion is periodic and nearly harmonic.
Phase space for a pendulum: angle $\theta$ and angular velocity $\omega$

Extended phase space for the driven pendulum: angle $\theta$, angular velocity $\omega$, time $t$

For small driving force amplitude, for example $F_0/m = 0.2$, the motion is periodic and nearly harmonic. The extended phase space plot looks like a spiral staircase.

The red stars indicate the state of the pendulum $(\theta, \omega)$ at time intervals one driving period $T_f$ apart.

The collection of these points is the “Poincaré section”.

Poincaré Section for $F_0/m = 0.2$
The damped and driven pendulum shows several qualitatively different types of behavior when the driving amplitude $F_0/m$ is increased.

1. Periodic with period $T_f$ (driving period) and nearly harmonic.
2. Periodic but not harmonic
   - periodic with period $T_f$
   - periodic with period $2T_f$
   - periodic with period $4T_f$
   - ...
3. Chaotic
4. Periodic but not harmonic
   - periodic with period $T_f$
   - periodic with period $2T_f$
   - periodic with period $4T_f$
   - ...
5. Chaotic

A system is chaotic, when a small difference in the initial conditions leads to exponentially increasing differences between the trajectories generated from the initial conditions.

**Poincare section:**
- 1 point
- 2 points
- 4 points
- ...

Chaos motion is deterministic and unpredictable.