Forced harmonic motion – the damped and driven harmonic oscillator

Restoring force: \( F_s = -kx \)
Damping force: \( F_d = -cv \)
Driving force: \( F_f = F_0 \cos \omega_f t \)
Net force: \( F_{\text{net}} = -kx - cv + F_0 \cos \omega_f t \)

Newton’s 2nd law:
\[
m \frac{d^2 x}{dt^2} = -kx - b \frac{dx}{dt} + F_0 \cos \omega_f t
\]

Define:
\[
\frac{k}{m} = \Omega^2 \quad \frac{c}{m} = 2\gamma
\]

Equation of motion, 2nd order differential equation:
\[
\ddot{x} + 2\gamma \dot{x} + \Omega^2 x = \frac{F_0}{m} \cos \omega_f t
\]

Important (angular) frequencies:
\[
\omega_f \quad \text{Driving frequency, driving period} \quad T_f = \frac{2\pi}{\omega_f}
\]
\[
\Omega = \sqrt{\frac{k}{m}} \quad \text{Natural frequency of the undamped oscillator}
\]
\[
\bar{\omega} = \sqrt{\Omega^2 - \gamma^2} \quad \text{Frequency of the underdamped oscillator}
\]

To find the steady state solution for \( x \), consider the driving force the real part of a complex driving force.

Find the complex solution of the new differential equation, the real and imaginary parts of the solution will solve the differential equation separately.

Try a solution of the form
\[
x(t) = A e^{i(\omega_f t - \phi)} = A e^{i\omega_f t} e^{-i\phi}
\]
and insert into the differential equation:
\[
\Rightarrow \dot{x}(t) = i\omega_f A e^{i\omega_f t} e^{-i\phi} \quad \Rightarrow \ddot{x}(t) = -\omega_f^2 A e^{i\omega_f t} e^{-i\phi}
\]
\[
\left( -\omega_f^2 + i2\gamma\omega_f + \Omega^2 \right) A e^{i\omega_f t} e^{-i\phi} = \frac{F_0}{m} e^{i\omega_f t}
\]
\[
\left( \Omega^2 - \omega_f^2 \right) A + i2\gamma\omega_f A = \frac{F_0}{m} \cos \phi + \frac{F_0}{m} \sin \phi
\]

Equate real and imaginary parts:
\[
\left( \Omega^2 - \omega_f^2 \right) A = \frac{F_0}{m} \cos \phi
\]
\[
2\gamma\omega_f A = \frac{F_0}{m} \sin \phi
\]

Take the ratio of the two equations to find an expression for the tangent of the phase angle. Square each of the equations and add to find an equation for the amplitude \( A \). See next pages for the results.
Amplitude of oscillations:

\[ A = \frac{F_0/m}{\sqrt{(\Omega^2 - \omega_f^2)^2 + (2\gamma\omega_f)^2}} \]

\[ \Omega = 1, \gamma = 0.2, F_0/m = 1 \]

Phase difference between oscillator and applied force:

\[ \tan \phi = \frac{2\gamma\omega_f}{\Omega^2 - \omega_f^2} \]

\[ \Omega = 1, \gamma = 0.2, F_0/m = 1 \]