Diffusion equation for the random walk

Random walk in one dimension

l = step length

τ = time for a single step

p = probability for a step to the right, $q = 1 - p$ is the probability for a step to the left

$P_N(m)$ = probability to find the walker at position $x = ml$ at time $t = N\tau$

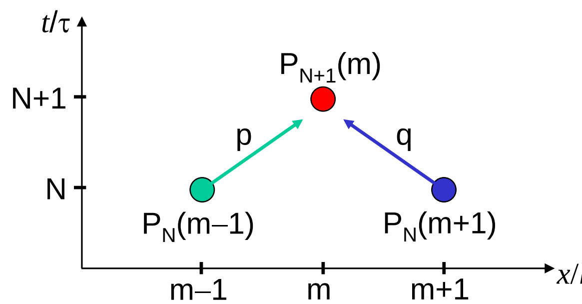
A random walk is a Markov process.

Let j and k be states (in this case positions) and let $p(j \to k)$ be the probability for a transition from j to k, then the transition probabilities

1. are independent of time
2. depend only on the states j and k, not on the history of the system
3. obey the sum rule $\sum_k P(j \to k) = 1$ (some state must be reached)

The probability $P_N(m)$ satisfies the stochastic difference equation

$$P_{N+1}(m) = pP_N(m-1) + qP_N(m+1)$$



Specialize to the case $p = q = 1/2$

$$P_{N+1}(m) = \frac{1}{2}P_N(m-1) + \frac{1}{2}P_N(m+1)$$

Subtract $P_N(m)$ on both sides

in the limit of large N, the differences become differentials

$$\underbrace{P_{N+1}(m) - P_N(m)}_{\cong \tau \frac{\partial P}{\partial t}} = \frac{1}{2}\underbrace{\left(P_N(m-1) + P_N(m+1) - 2P_N(m)\right)}_{\cong l^2 \frac{\partial^2 P}{\partial x^2}}$$

$$\frac{\partial P}{\partial t} = \underbrace{\frac{1}{2}\frac{l^2}{\tau}}_{\equiv D}\frac{\partial^2 P}{\partial x^2} \qquad D = \frac{l^2}{2\tau}$$

Diffusion equation $\quad \frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} \quad$ D is the (self) diffusion coefficient

Solve the diffusion equation with

boundary condition, for all times $\quad P(x,t) \to 0 \quad \text{as} \quad x \to \pm\infty$

initial condition (delta peak at the origin) $\quad P(x,0) = \delta(x)$

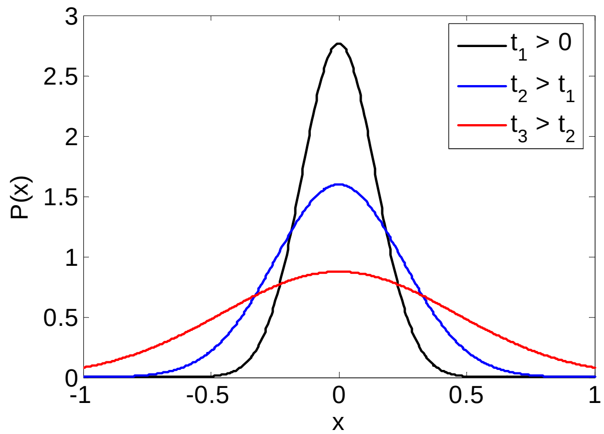
solution

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

with $\quad \sigma^2 = 2Dt$

For any time t $$P(x)dx = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) dx$$

As time goes on, the probability packet spreads



Mean square displacement (in one dimension) $$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} x^2 P(x) dx = 2Dt$$

in three dimensions one finds $$\left\langle r^2 \right\rangle = \left\langle x^2 + y^2 + z^2 \right\rangle = 6Dt$$