Damped harmonic motion

Newton’s 2nd law:
\[ m \frac{d^2 x}{dt^2} = -kx - \frac{dx}{dt} \]

Restoring force: \( F_s = -kx \)
Damping force: \( F_d = -cv \)
Net force: \( F_{\text{net}} = -kx - cv \)

Define:
\[ \ddot{x} = \frac{dx}{dt} \quad \dot{x} = \frac{d^2 x}{dt^2} \quad \frac{k}{m} = \Omega^2 \quad \frac{c}{m} = 2\gamma \]

Equation of motion, 2nd order differential equation:
\[ \ddot{x} + 2\gamma \dot{x} + \Omega^2 x = 0 \]

Solutions to the differential equation for \( x \):
\[ x(t) = A\exp(-(\gamma - a)t) + B\exp(-(\gamma + a)t) \]

Determine \( a \) by inserting the solution into the differential equation (\( A \) and \( B \) from initial conditions):
\[ \ddot{x}(t) = -(\gamma - a)A\exp(-(\gamma - a)t) - (\gamma + a)B\exp(-(\gamma + a)t) \]
\[ \ddot{x}(t) = (\gamma + a)^2 A\exp(-(\gamma - a)t) + (\gamma + a)^2 B\exp(-(\gamma + a)t) \]
\[ \Rightarrow \left( (\gamma - a)^2 - 2\gamma(\gamma - a) + \Omega^2 \right) A\exp(-(\gamma - a)t) + (\gamma + a)^2 (\gamma(\gamma - a) + \Omega^2) B\exp(-(\gamma + a)t) = 0 \]
\[ \Rightarrow a = +\sqrt{\gamma^2 - \Omega^2} \quad \Rightarrow a^2 = \gamma^2 - \Omega^2 \]

There are three types of solution, overdamped, underdamped, and critically damped.
(a) \( \gamma > \Omega \), overdamped

\[ \gamma > \Omega \Rightarrow a^2 > 0 \Rightarrow a = \sqrt{a^2} > 0 \quad a \text{ is a real, positive number} \]

\[ a = \sqrt{\gamma^2 - \Omega^2} = \gamma \sqrt{1 - \left( \frac{\Omega}{\gamma} \right)^2} < \gamma \]

- Both \( \exp(-(\gamma - a)t) \) and \( \exp(-(\gamma + a)t) \) decay exponentially
- \( \exp(-(\gamma + a)t) \) decays faster than \( \exp(-(\gamma - a)t) \)
- For long times, only the slowly decaying solution contributes.
(b) $\gamma < \Omega$, underdamped

$\gamma < \Omega \Rightarrow a^2 < 0 \Rightarrow a = \sqrt{\gamma^2 - \Omega^2} = i\sqrt{\Omega^2 - \gamma^2} = i\bar{a}$

$a$ is a purely imaginary number

$\bar{a}$ is a real positive number

$x(t) = A\exp(- (\gamma - a)t) + B\exp(- (\gamma + a)t)$

$= \exp(-\gamma t)(A\exp(i\bar{a}t) + B\exp(-i\bar{a}t))$

$= \exp(-\gamma t)(A'\cos(\bar{a}t) + B'\sin(\bar{a}t))$

Solutions are a product of exponential decay with time constant $1/\gamma$ and oscillating solutions with angular frequency:

$\bar{a} = \sqrt{\Omega^2 - \gamma^2}$

$\bar{a} = \Omega \sqrt{1 - \left(\frac{\gamma}{\Omega}\right)^2} < \Omega$

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graph showing exponential decay and oscillation.
\textbf{under damped} vs \textbf{exponential decay}}
\end{figure}

(c) $\gamma = \Omega$, critical damping

$\gamma = \Omega \Rightarrow a^2 = 0$

Start from:

$x(t) = \exp(-\gamma t)(A\exp(at) + B\exp(-at))$

Note that this is not equal to $\exp(-\gamma t)$.

Expand the exponential functions about $a = 0$:

$\exp(at) \approx 1 + at$

$\exp(-at) \approx 1 - at$

$x(t) = \exp(-\gamma t)((A + B) + (A - B)at)$

$= \exp(-\gamma t)(A' + B't)$

Solutions are a product of exponential decay with time constant $1/\gamma$ and a linear function in time:

$x(t) = \exp(-\gamma t)(A' + B't)$

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Graph showing critical damping and exponential decay.
\textbf{critical damping} vs \textbf{exponential decay}}
\end{figure}