Relativistic kinematics in one dimension – a particle subject to a constant force

A particle of mass \( m \) is moving along the \( x \) axis subject to a constant force \( F \).

The particle’s position, relativistic velocity, and momentum as a function of time \( t \) are \( x_r(t), v_r(t), \) and \( p(t) \), respectively.

In special relativity, the momentum and the velocity are related through

\[
\frac{p}{mc} = \sqrt{1 - \left(\frac{v_r}{c}\right)^2}
\]

From Newton’s 2\(^{nd} \) law in the form \( F = \frac{dp}{dt} \) we find with \( p_0 = p(t_0) \)

\[
p(t) = \int_{t_0}^{t} Fdt' + p_0 = F(t-t_0) + p_0 \Rightarrow F(t-t_0) + p_0 = \frac{m v_r}{\sqrt{1 - \left(\frac{v_r}{c}\right)^2}}
\]

To find the expression in the homework, solve for \( v_r \) and set \( v_{r_0} = 0 \Rightarrow p_0 = 0 \)

\[
\Rightarrow v_r = \frac{1}{m} \frac{F(t-t_0) + p_0}{\sqrt{1 + \left(\frac{F(t-t_0) + p_0}{mc}\right)^2}} = \frac{F(t-t_0)/m}{\sqrt{1 + \left(\frac{(t-t_0)F/m}{c}\right)^2}}
\]

Now integrate to find the position as a function of time for the case \( u_0 = 0 \).

\[
x_r(t) = \int_{t_0}^{t} v_r(t')dt' + x_0 = \frac{c^2}{F/m} \left( \sqrt{1 + \left(\frac{(t-t_0)F/m}{c}\right)^2} - 1 \right) + x_0
\]

How do we obtain the classical limit, \( x = \frac{1}{2} m \left( t-t_0 \right)^2 + x_0 \), from this equation?