1. Consider a 3-dimensional vector space with orthonormal basis. Let $A$, $B$ be operators represented in this basis by the following matrices:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix},$$

where $a$, $b$ are real numbers.

(a) Find the eigenvalues of $A$ and $B$ and note that both operators have degenerate spectra.

(b) Show that $[A, B] = 0$.

(c) Find a new set of orthonormal basis vectors which are simultaneous eigenkets of $A$ and $B$.

(d) Specify the eigenvalues of $A$ and $B$ for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

2. Prove the generalized uncertainty relation,

$$\langle\Delta \Omega\rangle^2 \langle\Delta \Lambda\rangle^2 \geq \frac{1}{4} |\langle[\Omega, \Lambda]\rangle|^2$$

where $\Omega$ and $\Lambda$ are any observables. Note, the statement is valid for any ket $|\Psi\rangle$, hence the subscript is dropped. You should use (and show where necessary) the following facts:

(a) The Schwarz inequality, valid for any two kets $|V\rangle$ and $|W\rangle$

$$\langle W|W\rangle \langle V|V\rangle \geq |\langle W|V\rangle|^2.$$  \hspace{1cm} (4)

(b) The expectation value of a Hermitean operator is a real number (show).

(c) The expectation value of an anti-Hermitean operator is a purely imaginary number (show).
(d) The operators
\[ \hat{\Omega} = \Omega - \langle \Omega \rangle \quad \text{and} \quad \hat{\Lambda} = \Lambda - \langle \Lambda \rangle \] (5)
are Hermitean (show).

(e) \[ [\hat{\Omega}, \hat{\Lambda}] = [\Omega, \Lambda] \] (show). (6)

(f) For any two operators \( X, Y \) we have (show)
\[ XY = \frac{1}{2}[X, Y] + \frac{1}{2}\{X, Y\} \] (7)

(g) The commutator of two observables is anti-Hermitean.

(h) The anti-commutator of two observables is Hermitean.

To prove the uncertainty relation (3) apply the Schwarz inequality (4) to
\[ |V\rangle = \hat{\Omega}|\Psi\rangle, \quad (8) \]
\[ |W\rangle = \hat{\Lambda}|\Psi\rangle, \quad (9) \]
where \(|\Psi\rangle\) is any ket. The left hand side of the inequality will almost immediately have the desired form; to obtain the right hand side use (b)-(h).

3. Shankar page 63, 1.10.2*

4. Shankar page 63, 1.10.3*