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LETTERS AND COMMENTS

Rolling of asymmetric discs on an inclined plane

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Abstract

In recent papers, Turner and Turner (2010 *Am. J. Phys.* **78** 905–7) and Jensen (2011 *Eur. J. Phys.* **32** 389–97) analysed the motion of asymmetric rolling rigid bodies on a horizontal plane. These papers addressed the common misconception that the instantaneous point of contact of the rolling body with the plane can be used to evaluate the angular momentum $\mathbf{L}$ and the torque $\boldsymbol{\tau}$ in the equation of motion $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$. To obtain the correct equation of motion, the 'phantom torque' or various rules that depend on the motion of the point about which $\mathbf{L}$ and $\boldsymbol{\tau}$ are evaluated were discussed. In this letter, I consider asymmetric discs rolling down an inclined plane and describe the most basic way of obtaining the correct equation of motion, that is, to choose the point about which $\mathbf{L}$ and $\boldsymbol{\tau}$ are evaluated that is stationary in an inertial frame.

In recent papers, L Turner and A M Turner [1] and Jensen [2] discussed the dynamics of a rolling body which is not azimuthally symmetric on a horizontal plane and the standard technique described in introductory physics textbooks for obtaining the equation of motion, which is to use $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$, where $\mathbf{L}$ is the angular momentum and $\boldsymbol{\tau}$ is the torque with respect to the contact point of the rolling body with the plane, gives incorrect results. They derived the correct equation of motion, either by introducing an additional term which they refer to as a phantom torque [1] or by using rules [2] that depend on the motion of the point about which $\mathbf{L}$ and $\boldsymbol{\tau}$ are evaluated.

The phantom torque in [1] was needed because the point about which the torques and the angular momenta were evaluated, the contact point between the hoop and the plane, was assumed to be attached to the semicircular hoop (see figure 2 in [1]), and hence it accelerates as the hoop rolls. The equation of motion $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$ is in general invalid when this point is accelerating with respect to an inertial frame, since it is based on Newton's second law which is valid only in an inertial frame. However, in the same way that the validity of Newton's second law can be restored in an accelerating (non-rotating) frame of reference by inclusion of a fictitious inertial force term, so can that of $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$ be with a phantom torque term as described in [1]. Indeed, as mentioned in [2], the phantom torque (equation (14) in [1]) can be obtained from the fictitious inertial force of an accelerated frame. Namely, in the frame of reference of a point P, the mass elements on an object, $\mathrm{d}m$, experience a fictitious force

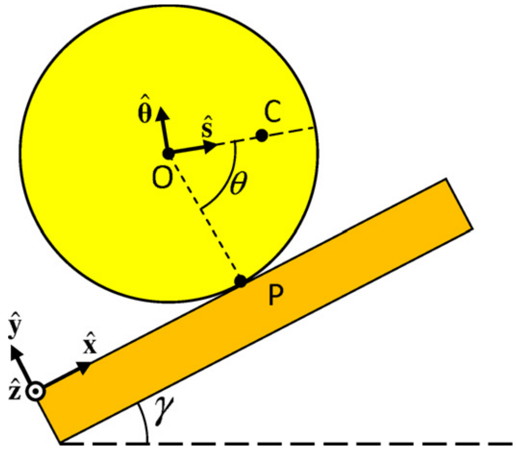

Figure 1. Rolling non-symmetric disc on a plane inclined at an angle γ to horizontal. O is the geometric centre, C is the centre of mass, P is the point of contact of the disc with the inclined plane and θ is the angle of the line through O and C with respect to perpendicular to the plane. Unit vector $\hat{\mathbf{s}}$ points from O to C, and $\hat{\boldsymbol{\theta}} = \mathrm{d}\hat{\mathbf{s}}/\mathrm{d}\theta$.

(This figure is in colour only in the electronic version)

$\mathrm{d}\mathbf{F}_{\mathrm{fict}} = -\mathbf{a}_P\,\mathrm{d}m$, where $\mathbf{a}_P$ is the acceleration of P with respect to an inertial frame [3]. This results in a phantom torque $\boldsymbol{\tau}_{\mathrm{ph}} = \int(\mathbf{r}-\mathbf{r}_P)\times\mathrm{d}\mathbf{F}_{\mathrm{fict}} = \int(\mathbf{r}-\mathbf{r}_P)\times(-\mathbf{a}_P\,\mathrm{d}m) = -M\mathbf{r}_{CP}\times\mathbf{a}_P$, where M is the total mass of the object and $\mathbf{r}_{CP}$ is the position of its centre of mass with respect to P, since $\int(\mathbf{r}-\mathbf{r}_P)\,\mathrm{d}m = M\mathbf{r}_{CP}$.

Reference [2] introduces two different rules to take into account the motion of the point of reference about which $\mathbf{L}$ and $\boldsymbol{\tau}$ are evaluated. The second rule given in equation (10) of [2] is equivalent to the phantom torque technique of [1].

This leads naturally to the question: why not simply solve the problem using the most basic method, that is, using the equation of motion $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$ where $\mathbf{L}$ and $\boldsymbol{\tau}$ are evaluated about a point P that is stationary with respect to an inertial frame? I consider this method in this letter, for the more general case than in [1] and [2], that of a planar disc of radius R and mass M rolling down an inclined plane, which is tilted at an angle γ relative to horizontal. The disc does not necessarily have uniform density, and has a centre of mass that is a distance βR from the geometric centre of the disc (where $0 \leqslant \beta < 1$). The friction of the inclined plane is sufficient to prevent the disc from slipping. Let θ be the angle of rotation of the cylinder in the counter-clockwise direction, and $\theta = 0$ correspond to the case where the centre of mass, C, the point of contact of the cylinder with the inclined plane, P, and the geometric centre of the cylinder, O, are co-linear. Define $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ to be the unit vectors in the directions of motion of the disc, perpendicular to the inclined plane, and perpendicular to the surface of the disc, respectively. Let the unit vector $\hat{\mathbf{s}}$ be in the direction from O to C, and $\hat{\boldsymbol{\theta}}$ be the unit vector perpendicular to $\hat{\mathbf{s}}$, as shown in figure 1. In this letter, I use double subscripts in variables to denote 'with respect to'. For example, $\mathbf{r}_{CP}$ is the position of C with respect to P.

The point P is assumed to be *attached to the plane on which the disc is rolling*, and hence is stationary with respect to an inertial frame. The angular momentum of the object with respect to a point P can be separated[1] into an 'orbital' part $\mathbf{L}_{\mathrm{orb}}$, the angular momentum of the

[1] See e.g. [3]: Taylor, p 370; Fowles and Cassiday, p 280.

centre of mass with respect to P, and a 'spin' part $\mathbf{L}_{\text{spin}}$, the angular momentum relative to the centre of mass of the object:

$$\mathbf{L} = \mathbf{L}_{\text{orb}} + \mathbf{L}_{\text{spin}} \tag{1a}$$
$$\mathbf{L}_{\text{orb}} = M\mathbf{r}_{CP} \times \mathbf{v}_{CP} \tag{1b}$$
$$\mathbf{L}_{\text{spin}} = \mathbb{I}_C \cdot \boldsymbol{\omega}, \tag{1c}$$

where $\mathbf{v}_{CP}$ is the velocity of the centre of mass with respect to P, $\mathbb{I}_C$ is the moment of inertia tensor relative to the centre of mass and $\boldsymbol{\omega}$ is the angular velocity vector. Using equations (1) in the equation of motion $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$, for the case considered here[2] gives

$$\mathbb{I}_C \cdot \dot{\boldsymbol{\omega}} + M\mathbf{r}_{CP} \times \mathbf{a}_C = \boldsymbol{\tau}, \tag{2}$$

where $\mathbf{a}_C$ is the acceleration of the centre of mass relative to an inertial frame.

The first term on the left-hand side of equation (2) is simply $I_C\ddot{\theta}\hat{\mathbf{z}}$, where I_C is the moment of inertia through C in the z-direction. To determine the second term, we need to find $\mathbf{a}_C$ as a function of θ. The position of the centre of mass C relative to the $\theta = 0$ point of the contact of the disc and the inclined plane is $R(-\theta\hat{\mathbf{x}} + \hat{\mathbf{y}} + \beta\hat{\mathbf{s}})$; i.e. the position of O relative to the point of contact for $\theta = 0$ plus the position of C relative to O. Taking the second derivative with respect to time, and using the result for the angular and centripetal accelerations for circular motion[3], gives $\mathbf{a}_C = R(-\ddot{\theta}\hat{\mathbf{x}} + \beta\ddot{\theta}\hat{\boldsymbol{\theta}} - \beta\dot{\theta}^2\hat{\mathbf{s}})$. This, together with $\mathbf{r}_{CP} = R(\hat{\mathbf{y}} + \beta\hat{\mathbf{s}})$, gives

$$\begin{aligned} M\mathbf{r}_{CP} \times \mathbf{a}_C &= MR^2(\hat{\mathbf{y}} + \beta\hat{\mathbf{s}}) \times (-\ddot{\theta}\hat{\mathbf{x}} + \beta\ddot{\theta}\hat{\boldsymbol{\theta}} - \beta\dot{\theta}^2\hat{\mathbf{s}}) \\ &= MR^2[(1 + \beta^2 - 2\beta\cos\theta)\ddot{\theta} + \beta\sin\theta\dot{\theta}^2]\hat{\mathbf{z}}, \end{aligned} \tag{3}$$

where $\hat{\mathbf{s}} = \sin\theta\,\hat{\mathbf{x}} - \cos\theta\,\hat{\mathbf{y}}$ and $\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}$ is used to evaluate the cross products. The direction of the weight force, in the case of an inclined plane that is tilted with an angle of γ, is $\mathbf{F}_{\text{wt}} = -Mg(\sin\gamma\,\hat{\mathbf{x}} + \cos\gamma\,\hat{\mathbf{y}})$ and therefore the torque with respect to P is

$$\boldsymbol{\tau} = \mathbf{r}_{CP} \times \mathbf{F}_{\text{wt}} = MgR\,[\sin\gamma - \beta\sin(\theta + \gamma)]\,\hat{\mathbf{z}}. \tag{4}$$

(The normal and frictional forces of the inclined plane on the disc act through point P, so do not contribute to the torque with respect to P.) Thus, equation (2) can be written as

$$I_P\ddot{\theta} + MR^2\beta\sin\theta\,\dot{\theta}^2 = MgR[\sin\gamma - \beta\sin(\theta + \gamma)], \tag{5}$$

where $I_P = I_C + M(1 + \beta^2 - 2\beta\cos\theta)R^2$ is the moment of inertia about the point P by the parallel axis theorem, since $(1 + \beta^2 - 2\beta\cos\theta)R^2 = |\mathbf{r}_{CP}|^2$. For the case of a uniform semicircular hoop on a level plane, where $\gamma = 0$, $\beta = 2/\pi$ and $I_P = 2MR^2\big(1 - \frac{2}{\pi}\cos\theta\big)$, equation (5) reproduces the results obtained in [1] and [2].

The approach typically adopted by introductory texts is to use $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$ where $\mathbf{L}$ and $\boldsymbol{\tau}$ are evaluated with respect to P. However, typically the approach incorrectly mixes two methods—the point P is assumed to be stationary in an inertial frame, but in evaluating $\mathrm{d}\mathbf{L}/\mathrm{d}t$, and the point P is assumed to be attached to the disc, giving in general the incorrect equation of motion $I_P\ddot{\theta} = \tau$. It misses the term $MR^2\beta\sin\theta\,\dot{\theta}^2$, which can be generated by including a phantom torque term that arises from the acceleration of the point P. When the point P is compelled to be stationary with respect to an inertial frame, the phantom torque vanishes. Nevertheless, the $MR^2\beta\sin\theta\,\dot{\theta}^2$ still appears, coming instead from the centripetal acceleration of C of the disc relative to O. This term vanishes when O and C are coincident, as in the standard case discussed in introductory physics textbooks, which is why this error has escaped detection. In a sense, the error is a subtle case of the standard freshman-level

[2] The $\boldsymbol{\omega} \times \mathbf{L}$ term in the Euler equation for $(\mathrm{d}\mathbf{L}/\mathrm{d}t)_{\text{space}}$ (see e.g. [3]: Taylor, p 396; Fowles and Cassiday, *op. cit.* p 382) for this system vanishes because $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$.

[3] See e.g. [3]: Taylor, p 29; Fowles and Cassiday, p 38.

mistake of confusing zero velocity with zero acceleration. Just because the point of contact between the disc and the plane is instantaneously stationary does not mean that it can be used as a point of reference for the equation of motion $\mathrm{d}\mathbf{L}/\mathrm{d}t = \boldsymbol{\tau}$. The point of reference must either have a zero acceleration with respect to an inertial frame, as in this letter, or if it has non-zero acceleration, as discussed in [1] and [2], additional terms must be introduced.

References

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[2] Jensen J H 2011 Rules for rolling as a rotation about the instantaneous point of contact *Eur. J. Phys.* **32** 389–97
[3] See e.g. Taylor J R 2005 *Classical Mechanics* (Sausalito, CA: University Science Books) p 328
Fowles G R and Cassiday G L 2005 *Analytical Mechanics* 7th edn (Belmont, CA: Brooks/Cole) p 185