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Introducing electromagnetic field momentum

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Abstract

I describe an elementary way of introducing electromagnetic field momentum. By considering a system of a long solenoid and line charge, the dependence of the field momentum on the electric and magnetic fields can be deduced. I obtain the electromagnetic angular momentum for a point charge and magnetic monopole pair partially through dimensional analysis and without using vector calculus identities or the need to evaluate integrals. I use this result to show that linear and angular momenta are conserved for a charge in the presence of a magnetic dipole when the dipole strength is changed.

(Some figures may appear in colour only in the online journal)

1. Introduction

The Feynman disc paradox [1, 2] is a striking illustration of momentum carried by electromagnetic fields. In this paradox, a ring of charges is placed on a rotating disc around a current carrying solenoid. The electric current is then turned off, causing the magnetic field to disappear. The Maxwell–Faraday law

$$\oint_{\mathcal{C}} \mathbf{E}_{\text{MF}} \cdot d\mathbf{r} = -\frac{\partial \Phi}{\partial t}, \quad (1)$$

where \mathbf{E}_{MF} is the induced (as in Lenz's law) electric field, \mathcal{C} is a closed contour and Φ is the magnetic flux through \mathcal{C} , implies that the change in the magnetic field induces an electric field in the space around the solenoid. The electric field imparts an impulse on the charge, which changes the angular momentum of the system, seemingly violating the conservation of angular momentum. This paradox is resolved when the angular momentum of the electromagnetic field around the solenoid is taken into account.

However, Feynman in his *Lectures in Physics* [1] did not quantitatively show that the angular momentum contained in the electromagnetic field is equal to the angular momentum imparted to the charge, perhaps because explicitly calculating the electromagnetic momentum is not easy [3–6]. Furthermore, students who first encounter this paradox typically attempt

erroneously to explain it by claiming that the angular momentum of the charge carriers which carry the current in the solenoid is transferred to the ring of charges. They reason that since the current in the solenoid must be varied to change the magnetic field, if one takes both the angular momentum of the ring of charges and the charges in the solenoid into consideration, angular momentum would be conserved and there would be no paradox. (This is not true, because the direction of the angular momentum imparted to the ring of charges depends on the sign of the charge, so if the angular momentum were conserved for one sign of charge, it would not be when the charges are reversed.)

There has been much less discussion in the literature of equivalent but simpler ‘paradoxes’ involving linear momentum. In one relatively early paper [7], Calkin used this paradox to derive the relationship between the electromagnetic linear momentum and the transverse vector potential. More recently, Cassenberg [8] described a method of introducing electromagnetic momentum by considering the discharging of a capacitor in a magnetic field and relating the impulse on current to the momentum stored in the field. This example is also used as a textbook exercise [9]. However, Babson *et al* [10] pointed out that this argument is ‘almost entirely wrong’ for very subtle reasons.

In this paper, I describe how electromagnetic momentum can be introduced in an elementary way by considering the linear momentum ‘paradox’ in a system consisting of an infinite solenoid and infinite line charge. Using this system, the dependence of the electromagnetic momentum density on the electric and magnetic fields can be deduced. I then describe a relatively simple method for obtaining the electromagnetic angular momentum of a magnetic monopole–point charge pair. The result is then utilized to demonstrate conservation of linear and angular momenta when the magnetic dipole moment is changed in the presence of a point charge, which is an idealized version of the original Feynman disc paradox. This paper thus provides a concise introduction to electromagnetic momentum and various associated phenomena.

The paper is organized as follows. Section 2 describes how the conservation of momentum in an infinite solenoid and line charge system can be used to deduce the existence of electromagnetic momentum and the dependence of electromagnetic momentum density on electric and magnetic fields. In section 3, the angular momentum of an electric charge–magnetic monopole pair is obtained, and this result is used to show that the linear and angular electromagnetic momenta of a point charge in the presence of a magnetic dipole is conserved. Section 4 contains a summary.

2. Deducing the existence of the electromagnetic momentum

The standard method for deriving the electromagnetic momentum is to first define an electromagnetic stress tensor $\overset{\leftrightarrow}{\mathbf{T}}$. Then, Maxwell’s equations and a series of vector identities are used to obtain [11] $\mathbf{f} + \epsilon_0\mu_0\partial\mathbf{S}/\partial t = \nabla \cdot \overset{\leftrightarrow}{\mathbf{T}}$, where \mathbf{f} is the force density (force per unit volume) and $\mathbf{S} = \mu_0^{-1}\mathbf{E} \times \mathbf{B}$ is the Poynting vector. This leads to the fact that $\epsilon_0\mu_0\mathbf{S} = \epsilon_0\mathbf{E} \times \mathbf{B} \equiv \mathbf{g}$ is the electromagnetic momentum density associated with an electromagnetic field. This derivation is not very physically enlightening, especially for a student encountering the concept of an electromagnetic momentum density for the first time. In this paper, I describe a more transparent way of introducing the concept of electromagnetic momentum, which is similar to but simpler than the Feynman disc paradox.

Consider an infinitely long straight thin wire with the uniform linear charge density λ that is parallel to the z -axis and that passes through a point $(R, 0, 0)$, and an infinitely long thin solenoid of very small circular cross-sectional area A centred on the z -axis, as shown in figure 1.

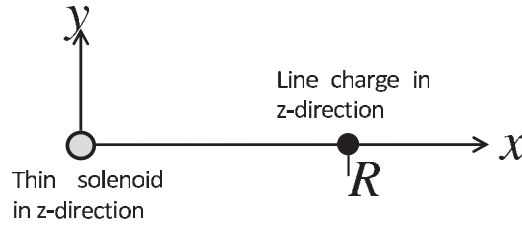


Figure 1. Configuration used to show the existence of electromagnetic momentum density \mathbf{g} . A thin solenoid with the small circular cross-sectional area A is centred along the z -axis, and a parallel line charge λ is at $x = R, y = 0$. When the magnetic field in the solenoid is changed, an electric field is generated around the solenoid that imparts an impulse on the line charge in the y -direction and which allows us obtain the dependence of \mathbf{g} on the electric and magnetic fields.

Assume that initially there is a uniform magnetic field $B\hat{\mathbf{z}}$ in the solenoid that is turned off in the time interval $t = 0$ to $t = t_f$, so that there is no magnetic field in the solenoid for $t \geq t_f$. The change in the magnetic flux $\Delta\Phi = -BA$ induces an electric field around the solenoid. By symmetry, the electric field is in the azimuthal direction and its spatial dependence of the magnitude depends only on s , the distance to the z -axis, i.e. $\mathbf{E}_{\text{MF}}(\mathbf{r}, t) = E_{\text{MF}}(s, t)\hat{\boldsymbol{\phi}}$, where the subscript ‘MF’ is used to distinguish it from the Coulomb’s law electric field due to the line charge at $x = R, y = 0$. For a contour C_R of a circle of radius R (with R larger than the radius of the solenoid) that is in the x - y plane and is centred at the origin, the line integral in the counterclockwise direction is

$$\oint_{C_R} \mathbf{E}_{\text{MF}}(\mathbf{r}, t) \cdot d\mathbf{r} = 2\pi R E_{\text{MF}}(R, t). \quad (2)$$

Substituting this into equation (1) gives

$$E_{\text{MF}}(R, t) = -\frac{1}{2\pi R} \frac{\partial\Phi}{\partial t}. \quad (3)$$

Integrating equation (3) over time from $t = 0$ to $t = t_f$ gives

$$\int_0^{t_f} E_{\text{MF}}(s, t) dt = -\frac{\Delta\Phi}{2\pi R} = \frac{BA}{2\pi R}. \quad (4)$$

Since the electric field is in the azimuthal direction, at the position of the line charge that is on the positive x -axis, the electric field is in the y -direction. The impulse per unit length $\vec{\mathcal{P}}_\lambda$ on the line charge λ at $x = R, y = 0$ due to \mathbf{E}_{MF} is

$$\vec{\mathcal{P}}_\lambda = \lambda \int_0^{t_f} \mathbf{E}_{\text{MF}}(t) dt = \frac{\lambda BA}{2\pi R} \hat{\mathbf{y}}. \quad (5)$$

The impulse imparts linear momentum on the line charge. Therefore, for linear momentum to be conserved, there must have been linear momentum in the system before the magnetic field was turned off.

If there were no line charge or no magnetic field in the solenoid, there would be no impulse on the line charge and hence no linear momentum would be imparted to the line charge. This suggests that the electromagnetic momentum density is in regions of space, which contain *both* electric and magnetic fields. The only region of space that contains both electric and magnetic fields is within the thin solenoid along the z -axis.

The electric field due to the line charge along the z -axis (i.e. inside the solenoid) is $\mathbf{E} = E\hat{\mathbf{x}}$, where $E = -\lambda/(2\pi\epsilon_0 R)$. Therefore, the momentum per unit length contained in the electromagnetic field given by equation (5) can be written as $-\epsilon_0 EBA\hat{\mathbf{y}}$. Hence, the

electromagnetic momentum density (i.e. per unit volume), which is the momentum per unit length divided by the area, is $\mathbf{g} = -\epsilon_0 EB\hat{\mathbf{y}}$. Assuming that the electromagnetic momentum density is a local function of both the electric and the magnetic fields $\mathbf{g}(\mathbf{E}, \mathbf{B})$, the only way to produce a vector $\mathbf{g} = -\epsilon_0 EB\hat{\mathbf{y}}$ out of the vectors $\mathbf{B} = B\hat{\mathbf{z}}$ and $\mathbf{E} = E\hat{\mathbf{x}}$ is

$$\mathbf{g}(\mathbf{E}, \mathbf{B}) = \epsilon_0 \mathbf{E} \times \mathbf{B}. \quad (6)$$

Hence, the conservation of linear momentum and the assumption that the electromagnetic momentum density depends on local electric and magnetic fields imply that $\mathbf{g}(\mathbf{E}, \mathbf{B})$ must have the form given in equation (6).

2.1. Consistency check

In order to check the consistency of this result, instead of a line charge, we can perform an analogous calculation for a single point charge Q at $\mathbf{r} = (R, 0, 0)$. If the field of the solenoid is changed from $B\hat{\mathbf{z}}$ to zero, the induced electric field imparts momentum

$$\mathbf{P}_Q = \frac{QBA}{2\pi R}\hat{\mathbf{y}} \quad (7)$$

on the charge Q .

The electric field as a function of position inside the solenoid along the z -axis due to the point charge is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0(R^2 + z^2)^{3/2}}(-R\hat{\mathbf{x}} + z\hat{\mathbf{z}}), \quad (8)$$

and therefore, the momentum density within the solenoid is

$$\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B} = \frac{QB}{4\pi} \frac{R}{(z^2 + R^2)^{3/2}}\hat{\mathbf{y}}. \quad (9)$$

Integrating the momentum density of the magnetic field within the solenoid gives the total electromagnetic momentum of

$$\mathbf{P}_{\text{EM}} = \frac{QAB}{4\pi} \int_{-\infty}^{\infty} dz \frac{R}{(R^2 + z^2)^{3/2}}\hat{\mathbf{y}} = \frac{QAB}{2\pi R}\hat{\mathbf{y}}. \quad (10)$$

Thus, the momentum is conserved since $\mathbf{P}_Q = \mathbf{P}_{\text{EM}}$.

2.2. Hidden momentum

There is a subtle effect, referred to as the hidden momentum, associated with the motion of a current in a loop in the presence of a potential field [12, 13]. It is a purely kinematic relativistic effect and has nothing to do with electromagnetism. The presence of hidden momentum is necessitated by a theorem that states that in a static system, the centre of energy is stationary, and hence, the total momentum must be zero [13, 14]. Hence, when there is a static magnetic field in the solenoid, the hidden momentum per unit length due to the current in the solenoid is equal in magnitude and opposite in direction to the electromagnetic momentum inside the solenoid.

The physical explanation of the hidden momentum is as follows. Assume for simplicity that the current in the solenoid is caused by positive charges, and the line charge at $x = R, y = 0$ is positive. For these choices of signs of charges, the electric potential energy of the current-carrying charges on the far side of the solenoid from the line charge ($x < 0$ in figure 1) is lower than the near side ($x > 0$ in figure 1). By conservation of energy, the kinetic energy and hence the average speed of the current-carrying charges are larger on the far side than on the near side.

The magnitude of the current density in the solenoid $j = \lambda_s v$, where λ_s is the current-carrying charge density in the solenoid and v is the average speed. In a steady-state situation, j in the solenoid must be constant. This implies that λ_s on the far side of the solenoid will be slightly less than on the near side, because of the difference in the average speed of the charges. If the momentum were strictly proportional to the velocity, then the magnitude of the momentum would be strictly proportional to the current and the momentum density would also be constant in the solenoid. However, the momentum is *not* proportional to velocity, but is in fact $\mathbf{P} = \gamma m \mathbf{v}$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. This results in the magnitude momentum on the far side of the solenoid being *larger* than the magnitude of the momentum on the near side, resulting in a non-zero net mechanical momentum of the current-carrying charges in the solenoid. It can be shown [10, 12] that in the case of closed current loops, the hidden momentum is equal to $\mathbf{P}_{\text{hid}} = c^{-2} \mathbf{m} \times \mathbf{E}$ (where \mathbf{m} is the magnetic moment of an infinitesimal current loop), or equivalently the hidden momentum density per unit volume is $\mathbf{p}_{\text{hid}} = c^{-2} \mathbf{M} \times \mathbf{E}$, where \mathbf{M} is the magnetic moment per unit volume, or magnetization. A solenoid with uniform field \mathbf{B} can be obtained by having magnetization $\mathbf{M} = \mathbf{B}/\mu_0$ inside the solenoid, so the hidden momentum per unit volume inside the solenoid is $\vec{\mathcal{P}}_{\text{hid}} = c^{-2} \mu_0^{-1} \mathbf{B} \times \mathbf{E} = \epsilon_0 \mathbf{B} \times \mathbf{E}$ (since $c^2 \epsilon_0 \mu_0 = 1$), which is exactly opposite to the electromagnetic momentum density.

Does the presence of the hidden momentum affect the argument given above for the existence of electromagnetic momentum? The answer is ‘no’. When hidden momentum is taken into account, the total momentum per unit length of the system is zero, because the hidden momentum and the electromagnetic momentum cancel each other (as required by the stationary centre of energy theorem [14]). As the current in the solenoid is turned off, the magnetic field in the solenoid tends to zero, as does the electromagnetic momentum in the solenoid. As described in the beginning of this section, this momentum is transferred to the line charge by the induced Maxwell–Faraday electric field \mathbf{E}_{MF} . On the other hand, the hidden momentum, being purely mechanical in origin, is transferred to the structure of the solenoid and remains within the solenoid when the solenoid current is turned off. Thus, the total momentum of the system is still zero, i.e. is conserved, since the line charge and the solenoid have equal and opposite momenta per unit length; the line charge picks up the momentum that was originally in the electromagnetic field, and the structure of the solenoid picks up the momentum that was originally in the hidden momentum.

Note that the hidden momentum is not at all equivalent to the erroneous student argument to explain the angular momentum paradox mentioned in section 1. The hidden momentum is the *linear* momentum caused by a circulating current in the presence of a potential (in this case, electric) field and is a purely relativistic effect, while the erroneous student argument is an attempt to explain the Feynman paradox by ascribing *angular* momentum to the circular motion of the charges around the solenoid and is a purely non-relativistic argument.

3. Angular momentum due to point electric charge and magnetic monopole

Using $\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$, we now derive the electromagnetic field angular momentum for a point charge Q and magnetic monopole q_m pair is [15–17]

$$\mathbf{L} = \frac{Qq_m}{4\pi} \frac{\mathbf{R}}{R}, \quad (11)$$

where \mathbf{R} is displacement of the magnetic monopole from the dipole, without using vector calculus or evaluating any integrals. This important result was used by Dirac to argue that if a single magnetic monopole were observed, then electric charge must be quantized [17]. We subsequently use this to derive quantitatively the linear and angular momenta due to a point charge in the presence of a magnetic dipole.

A magnetic monopole is a (as yet unobserved) source of magnetic field, which is analogous to that of a point charge for an electric field. The fields at position \mathbf{r} for the electric and magnetic monopoles at the origin have the form

$$\mathbf{E}_Q(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}, \quad (12a)$$

$$\mathbf{B}_{q_m}(\mathbf{r}) = \frac{q_m}{4\pi r^2} \hat{\mathbf{r}}, \quad (12b)$$

where $r = |\mathbf{r}|$.

This result can be derived to within a dimensionless factor from dimensional analysis. First, note that the electromagnetic angular momentum of the a point charge and magnetic monopole pair is independent of the origin, because the total linear momentum of the system is $\mathbf{P}_{\text{em}} = \int \mathbf{g} d^3x = 0$.¹ The vanishing of \mathbf{P}_{em} can be deduced from the fact that $\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$ points in the azimuthal direction and is azimuthally symmetric with respect to the \mathbf{R} direction.²

Since \mathbf{L}_{em} for the point charge–magnetic monopole pair is independent of origin, let us choose the origin to be at the position of the point charge. Using equation (12), we obtain

$$\begin{aligned} \mathbf{L}_{\text{em}} &= \int d^3x \mathbf{r} \times \mathbf{g} = \epsilon_0 \int d^3x \mathbf{r} \times (\mathbf{E}_Q(\mathbf{r}) \times \mathbf{B}_{q_m}(\mathbf{r} - \mathbf{R})) \\ &= Qq_m \left[\int d^3x \frac{1}{(4\pi)^2} \frac{\mathbf{r} \times (\mathbf{r} \times (\mathbf{r} - \mathbf{R}))}{r^3 |\mathbf{R} - \mathbf{r}|^3} \right]. \end{aligned} \quad (13)$$

Note that Qq_m has units of angular momentum. The integral in the square brackets in equation (13) is dimensionless and is a vector quantity that depends only on \mathbf{R} , since the variable \mathbf{r} is integrated over.³ The only function that meets these conditions is $C\mathbf{R}/R$, where C is as yet an undetermined dimensionless constant. This implies that

$$\mathbf{L}_{\text{em}} = CQq_m \frac{\mathbf{R}}{R}, \quad (14)$$

where C is determined next.

3.1. Determining C

Consider a magnetic monopole q_m at the origin, and two circular capacitor plates of area \mathcal{A} , which are oriented parallel to the x - y plane with their axes along the z -axis, at $z = \pm l$, where $l \ll \sqrt{\mathcal{A}}$, as shown in figure 2. The plate at $z = +l$ has the charge density $-\sigma$ and the one at $z = -l$ has the charge density $+\sigma$. We can deduce C by writing the electromagnetic angular momentum of the magnetic monopole in between the capacitor plates in two different ways, and comparing the results.

Method 1: $\mathbf{L}_{\text{em}} = \epsilon_0 \int d^3x \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$. The electric field due to the charged plates is in the $+\hat{\mathbf{z}}$ direction. Within the closely separated capacitor plates, except for a negligibly small volume near the z -axis, the \mathbf{B} -field due to the magnetic monopole points essentially radially outwards from the z -axis; therefore, in between the capacitor plates, $\mathbf{E} \times \mathbf{B}$ is in the azimuthal direction and $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ is in the $\hat{\mathbf{z}}$ -direction. Since \mathbf{E} and \mathbf{B} are essentially perpendicular, and \mathbf{r} and $\mathbf{E} \times \mathbf{B}$ are essentially perpendicular, $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ has the magnitude

¹ This is because a shift in the origin \mathbf{r}_0 changes the angular momentum by $\int d^3x \mathbf{r}_0 \times \mathbf{g} = \mathbf{r}_0 \times \mathbf{P}_{\text{em}}$.

² An alternative argument for $\mathbf{P}_{\text{em}} = 0$ is given in [17]. Because \mathbf{P} is a vector, and since the only relevant vector in this case is \mathbf{R} , it must be that $\mathbf{P} \propto \mathbf{R}$. However, the integrand $\mathbf{E}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})$ for all points \mathbf{r} is always perpendicular to \mathbf{R} because the displacement vectors from the charge and magnetic monopole to \mathbf{r} are in the same plane as \mathbf{R} .

³ The integral does not diverge at $\mathbf{r} = \mathbf{0}$ and $\mathbf{r} = \mathbf{R}$ because of phase space factors $r^2 dr$ and $r'^2 dr'$ (where $\mathbf{r}' = \mathbf{r} - \mathbf{R}$), respectively. At $r = \infty$, it also converges because the integrand goes to $1/r^4$ in this limit.

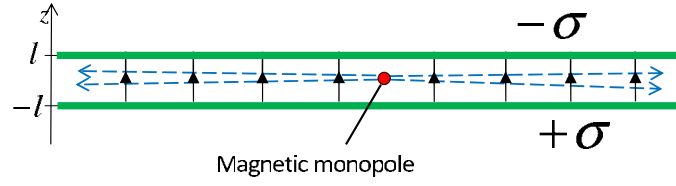


Figure 2. Magnetic monopole in the middle of a capacitor with closely separated circular plates. The solid lines with arrowheads are electric field lines and the broken lines are magnetic field lines. Except for a small and negligible volume around the magnetic monopole, the magnetic and electric field lines are almost perpendicular to each other. By equating the electromagnetic angular momentum evaluated by $\epsilon_0 \int d^3x \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ and by $\mathbf{L}_{\text{em}} = C\mathbf{R}/R$ for a magnetic monopole–point charge pair, the value of the unknown coefficient C is obtained.

rEB . The electric field in between capacitor plates of charge density σ is $E = \sigma/\epsilon_0$ and the magnetic field due to the monopole is $B = q_m/4\pi r^2$. Therefore,

$$\mathbf{L}_{\text{em}} = \epsilon_0 \int_{\mathcal{V}} d^3x rEB \hat{\mathbf{z}} = 2l\epsilon_0 \hat{\mathbf{z}} \int_{\mathcal{A}} d^2x r \frac{\sigma}{\epsilon_0} \frac{q_m}{4\pi r^2} = \frac{2lq_m\sigma}{4\pi} \hat{\mathbf{z}} \int_{\mathcal{A}} d^2x \frac{1}{r}, \quad (15)$$

where \mathcal{V} is the volume between the capacitor plates and \mathcal{A} is the area of one capacitor plate.

Method 2: \mathbf{L}_{em} using equation (14). The second way to evaluate the angular momentum is to consider the plate charges to be point charges spread uniformly over the plates. The \mathbf{L}_{em} is an integral of the angular momentum due to the pairing of every charge element on the plates with the magnetic monopole, which by equation (14) would be

$$\mathbf{L}_{\text{em}} = q_m C \int dQ \frac{\mathbf{R}}{R}, \quad (16)$$

where \mathbf{R} is the vector from the charge element dQ to the magnetic monopole. The electromagnetic angular momentum due to the charged plates and the magnetic monopole is

$$\mathbf{L}_{\text{em},\pm} = q_m C \sigma \int_{\mathcal{A}} d^2x \frac{\mathbf{r} \pm l\hat{\mathbf{z}}}{|\mathbf{r} \pm l\hat{\mathbf{z}}|}, \quad (17)$$

where ‘+’ and ‘−’ correspond to the positive and negative charged plates, respectively. Thus, the total field angular momentum, using $|\mathbf{r} \pm l\hat{\mathbf{z}}| \approx r$ (since $r \gg l$ for most of the integration over \mathcal{A}) is

$$\mathbf{L}_{\text{em}} = \mathbf{L}_{\text{em},+} + \mathbf{L}_{\text{em},-} = q_m C \sigma \int_{\mathcal{A}} d^2x \frac{\mathbf{r} + l\hat{\mathbf{z}}}{|\mathbf{r} + l\hat{\mathbf{z}}|} - \frac{\mathbf{r} - l\hat{\mathbf{z}}}{|\mathbf{r} - l\hat{\mathbf{z}}|} \approx 2lq_m\sigma C \hat{\mathbf{z}} \int_{\mathcal{A}} d^2x \frac{1}{r}. \quad (18)$$

Comparing equations (18) and (15) gives $C = (4\pi)^{-1}$, yielding equation (11).

3.2. Using this result to show the conservation of angular and linear momentum of a magnetic dipole

The result in equation (11) can be used to show quantitatively what Feynman did not in his textbook—when the magnetic dipole changes in the presence of a point charge, the linear and angular momenta are conserved. Let us assume that there is a point magnetic dipole \mathbf{m} at the origin and a point charge Q at \mathbf{r} . It can now be shown relatively easily that the linear and angular momenta imparted to the charge are equal to the momenta stored in the electromagnetic fields.

The magnetic field of a point magnetic dipole \mathbf{m} at \mathbf{r}_0 is identical to the magnetic field produced by a pair of magnetic monopoles q_m at $\mathbf{r}_0 + \mathbf{d}/2$ and $-q_m$ at $\mathbf{r}_0 - \mathbf{d}/2$, where

$\lim_{|d| \rightarrow 0} q_m \mathbf{d} \rightarrow \mu_0 \mathbf{m}$, plus a ‘contact term’ $\mathbf{B}_{\text{cont}}(\mathbf{r}) = \mu_0 \mathbf{m} \delta(\mathbf{r} - \mathbf{r}_0)$ [18]. The contact term ensures that the Maxwell equation $\nabla \cdot \mathbf{B} = 0$ is satisfied.

The linear momentum of this system is the sum of the linear momenta of the three magnetic ‘sources’, i.e. the two magnetic monopoles and the contact term, in the presence of the point charge. As noted earlier, the net linear momentum due to the magnetic monopoles and the point charge is zero. Therefore, the linear momentum is solely due to the contact term, which easily evaluated because \mathbf{B}_{cont} is a δ -function, giving

$$\mathbf{P}_{\text{em}} = \epsilon_0 \int \mathbf{E} \times \mathbf{B}_{\text{cont}}(\mathbf{r}) = \epsilon_0 \mu_0 \mathbf{E}(\mathbf{r} = \mathbf{0}) \times \mathbf{m} = \frac{\mu_0 Q}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \quad (19)$$

Lawson [3] showed using this model that for a magnetic dipole at the origin and a point charge q at position \mathbf{r} , the angular momentum (with respect to the origin) is $\mathbf{L}_{\text{em}} = \frac{\mu_0 q}{4\pi} \left(\frac{\mathbf{m}}{r} - \frac{\mathbf{r}(\mathbf{m} \cdot \mathbf{r})}{r^3} \right)$. Using the identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ gives

$$\mathbf{L}_{\text{em}} = \frac{\mu_0 q}{4\pi} \frac{\mathbf{r} \times (\mathbf{m} \times \mathbf{r})}{r^3}. \quad (20)$$

Equations (19) and (20) agree with the results of Griffiths [18], which were obtained by a lengthier but more general derivation.

To determine the impulse of the electric field on the charge Q when the magnetic moment is turned off, we use the fact that the electric field at \mathbf{r} due to arbitrary variation $\mathbf{m}(t)$ is [19, 20]

$$\mathbf{E}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \left[\mathbf{r} \times \left(\frac{\dot{\mathbf{m}}}{r^3} + \frac{\ddot{\mathbf{m}}}{r^2 c} \right) \right]_{\text{ret}}, \quad (21)$$

where ‘ret’ means that \mathbf{m} is evaluated at retarded time $t_{\text{ret}} = t - r/c$. Thus, $\int_{-\infty}^{\infty} \mathbf{E}(t) dt = \frac{\mu_0}{4\pi} \frac{\mathbf{r} \times \Delta \mathbf{m}}{r^3}$. Since the dipole is turned off, $\Delta \mathbf{m} = -\mathbf{m}$, and therefore, the impulse given to the charge Q is

$$\mathbf{P}_Q = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \quad (22)$$

and the angular momentum (with respect to the magnetic dipole) imparted to the charge Q is $\mathbf{r} \times \mathbf{P}_Q$. These results match the linear and angular momenta given in equations (19) and (20).⁴

4. Summary

This paper describes a concise introduction to electromagnetic momentum that is at the level of Feynman’s *Lectures in Physics*. By examining a system consisting of a long thin solenoid and line charge, the dependence on the electromagnetic momentum density on the electric and magnetic fields can be deduced. The angular momentum for a magnetic monopole–point charge pair is obtained partly through dimensional analysis, and without use of vector calculus identities or evaluation of integrals. The conservation of linear and angular momenta of charges in the presence of a changing magnetic dipole, an idealized version of the Feynman disc paradox, is explicitly demonstrated.

⁴ An alternative way of deriving this result is to note that changing the magnetic dipole induces an electric field \mathbf{E}_{MF} , which imparts an impulse $\mathbf{P}_Q = Q \int dt \mathbf{E}_{\text{MF}}(\mathbf{r}, t)$ on the charge Q at \mathbf{r} . The electromagnetic fields due to the dipole can be represented by a vector potential $\mathbf{A}(\mathbf{r}, t)$ in the Coulomb ($\nabla \cdot \mathbf{A} = 0$) gauge, in which case the scalar potential caused by the changing magnetic dipole is zero. Hence, the electric field is $\mathbf{E}_{\text{MF}} = -\partial \mathbf{A} / \partial t$ and the impulse on the charge q is [7] $\mathbf{P}_Q = -Q \int dt \frac{\partial \mathbf{A}}{\partial t} = -Q \Delta \mathbf{A}$. This, together with the vector potential for a magnetic dipole in the Coulomb gauge, $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}$, reproduces equation (22) when the magnetic moment \mathbf{m} is turned off.

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